European University Institute

DEPARTMENT OF ECONOMICS

Essays on Contests and Conflicts

Anastasia Antsygina

Thesis submitted for assessment with a view to obtaining the degree of Doctor of Economics of the European University Institute

Florence, 15 September 2017

European University Institute
Department of Economics

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Abstract

This thesis focuses on contests and conflicts with heterogeneous participants where the asymmetries arise from players' preferences, skills, or resource constraints. The work consists of two chapters. Each section proposes a theoretical framework and provides an empirical justification of the key patterns discovered.

In the first chapter, I develop a model where two players with asymmetric preferences engage in a contest game. Prizes consist of two non-tradable divisible goods. I characterize the optimal prize allocation that maximizes aggregate effort of the contestants. When heterogeneity is severe, the designer benefits by giving a positive prize to the loser. This allocation eliminates the advantage of the stronger competitor and makes the contest homogeneous. As a consequence, the opponent has higher chances to win and exerts more effort in equilibrium. This positive response increases aggregate effort.

The model mirrors the job promotion setting with monetary and non-monetary rewards. Using data from first-round matches of two professional tennis competitions where prizes include money and the ATP ranking points (career concerns), I structurally estimate contestants' skills and preferences. Overlooking multi-dimensionality results in biased estimates of the prize incentive effects. Counterfactual experiments show that reallocating 5% of money and 2% of the ATP ranking points from final winning prizes to first-round losing rewards could improve expected aggregate effort in relatively heterogeneous matches by more than 4.9%.

The second chapter, jointly written with Madina Kurmangaliyeva, adopts the contest setting to study main determinants of "victim-defendant" settlements. This institute is widely used in both civil and criminal legal practices. Understanding how the power imbalance affects the negotiation process is crucial for the optimal design of the justice system. We develop a theoretical model where the victim (she) and the defendant (he) must exert costly effort for the case to reach / avoid the court. Before the game starts, the defendant can settle with the victim by making her a "take-it-or-leave-it" offer. Improving the defendant's bargaining position reduces the settlement amount. When the victim has strong preferences for revenge, the agreement may fail to happen even if the defendant can afford the optimal offer.

Using the data on criminal traffic offenses in Russia for 2013–2014 (56'000 complete cases), we structurally estimate the model and recover individual preferences and fighting abilities. Overall, defendants are 10 times wealthier than their victims. Offenders who manage to settle face significantly less disutility than their peers going to court (–.007 against –810.78 for "car vs. pedestrian" accidents). Thus, "victim-defendant" settlements can rise the inequality before the law. If Russia abandoned the given institute, the prison population could increase by 2'850 inmates, which would cost the state €2.3 million per year.

Acknowledgements

First of all, I would like to thank my supervisor Prof. Piero Gottardi for his enormous support and extreme patience. Piero is a great scholar and an outstanding mentor who never forced me to go the way I did not want to. Also, my work particularly benefited from the comments of Prof. Andrea Mattozzi who was always approachable, supportive, and willing to help.

I am very grateful to Prof. Daniel Quint who helped me revive one of the projects when it was almost dead and also gave a chance to visit the University of Wisconsin-Madison (that was a great experience!). Finally, this work would be hard to complete without Prof. Daniela Iorio. I thank Daniela for her time devoted to me and all the comments she gave.

Many thanks go to Madina Kurmangaliyeva who first became my good friend and then turned to be a co-author. With her support, it was much easier to overcome the moments of frustration and much nicer to share the joy of small achievements.

I would like to thank Jessica Spataro, Anne Banks, Lucia Vigna, and Sarah Simonsen for their outstanding assistance and the willingness to help with any issues.

My Florentine experience would not be the same without Caroline Mezger, Olga Pavlenko, Kristina Bluwstein, Eric Radish, Irina Kirysheva, Katerina Yanke, Sergey Balakin, and my amazing classmates.

A special acknowledgement goes to my mother Nadezhda, whose support is priceless, and to my best friend Sasha Z.

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Chapter 1

Optimal Allocation of Multi-Dimensional Prizes in Contests with Heterogeneous Agents: Theory and an Empirical Application

1.1 Introduction

Many interactions can be characterized as bilateral contests with heterogeneous players and prizes consisting of multiple goods. The most important example is job promotion. Managers, who care about aggregate effort, can use not only monetary but also symbolic awards to motivate workers. The latter element of the prize schedule includes status or titles such as "Employee of The Month". Often companies allocate firm-specific rights and privileges in the contest setting. By definition, there is no way to express non-monetary benefits in monetary terms. Thus, contest managers allocate bundles of prizes.

Workers can have heterogeneous preferences over reward dimensions. Some individuals may hold stronger career concerns but value monetary benefits less than their peers. As a result, the same prize schedule can shape contestants' incentives differently. Managers must take this aspect into account when they allocate goods between bundles.

Overall, heterogeneity can worsen the contest performance significantly.[1] When the competition is uneven, participants have fewer incentives to exert effort. Disadvantaged contestants know that most likely they will lose, and do not compete much. On the other hand, stronger opponents can make less effort than they would exert in the homogeneous contest, but still win. If the manager is not able to pre-select competitors, he may end up with very uneven matches and, consequently, low aggregate effort. In this respect, it is important to find a way to mitigate the detrimental effect heterogeneity has on players' incentives.

Obviously, the contest needs sufficient winning prizes that provide enough stimuli to compete. At the same time, however, positive rewards for the weakest performers are not rare. For example, non-zero losing prizes are widespread in professional sports, which do not differ much from the job promotion setting. Each tournament of the Grand Slam series, a highly prestigious tennis competition, spends more than 10% of its total budget (about $4.5 million in 2015) to reward first-round losers.[2] Perhaps surprisingly, also in this case prizes consist of two goods: money and the ATP ranking points (career concerns).

Based on these considerations, I study the optimal prize allocation in contests with heterogeneous participants and multiple reward items. In particular, I propose a theoretical model with two players characterized by asymmetric commonly known preferences. The designer has endowments of two non-tradable divisible goods. He can allocate the items between winning and losing bundles to maximize the total effort.

[1] For example, see Lazear and Rosen (1981), Che and Gale (1998), Brown (2011).

[2] This series includes four competitions: the Australian Open, the French Open, the Wimbledon Championships, and the U.S. Open. The amount spent for first-round losing prizes exceeds the final winning prize by more than 50%.

When heterogeneity is strong, the optimal reward schedule prescribes a positive losing prize in the dimension valued by the advantaged participant relatively more. This scheme equalizes players' winning benefits and makes the contest homogeneous. In other words, the designer reduces the advantage of the strong player, and the chances of the weaker contestant winning increase. As a result, the latter participant exerts more effort than he would in case of the "Winner–Takes–All" prize allocation. This positive effect is indirect and works through the equilibrium interaction (equilibrium effect). If heterogeneity is severe, higher losing prizes reduce the winning benefit of the advantaged player significantly, but do not affect his opponent much. Then the positive equilibrium effect dominates effort losses associated with lower winning benefits. Consequently, the total effort increases. This result contrasts with the previous findings of the contest design literature.

The described mechanism never works in the one-dimensional setting. To make the competition homogeneous, the designer must assign identical single-item prizes for a winner and a loser. The allocation, however, destroys the contest: both participants exert zero effort. In the two-dimensional case, the designer not only makes the competition even but also provides strictly positive winning benefits. The result is robust to heterogeneity in skills and asymmetric information about players' types.

To stress the relevance of multi-dimensional prizes in job promotion and other similar contest settings, I collect data from first-round matches of two professional tennis competitions, the Australian Open and the French Open. The application is very close to the job promotion context. In particular, the athletes have two-dimensional incentives shaped by monetary prizes and career concerns (the ATP ranking points). At the same time, they show enough heterogeneity in skills and preferences. First-round winning prizes are approximated as a continuation value of being advanced in the tournament.[3] I structurally estimate players' skills and valuations. Importantly, both prize items matter for individual effort choices. When the points dimension is neglected, the model overestimates incentive effects of monetary prizes and underestimates contestants' heterogeneity. This finding reveals the direction to improve the existing reduced-form tests of the contest theory that, to the best of my knowledge, always overlook the multi-dimensionality aspect.[4]

Finally, I run counterfactual experiments and show how alternative reward schemes can improve aggregate effort in both tournaments. On average, the French Open features more heterogeneous matches and would benefit from larger than actual monetary losing prizes. However, the Australian Open could achieve the highest mean expected aggregate effort with zero monetary losing prizes. The observed difference can be explained with the contest-specific matching policy. If the managers

[3] Grand Slam events are single-elimination tournaments consisting of seven stages: four rounds, a quarterfinal, a semifinal, and a final.

[4] For example, see Maloney and McCormick (2000), Sunde (2009), Brown (2013), Berger and Nicken (2014).

of analyzed tournaments could change the ranking points allocation, they would prefer zero losing prizes in this good. At the match-specific level, when the competition is relatively heterogeneous, reducing final winning trophies in money and the ATP ranking points in favor of first-round losing rewards by 5% and 2%, respectively, could increase expected aggregate effort by more than 4.9%. Compared to the "Winner–Takes–All" allocation, the gain would exceed 5.6%. On the other hand, relatively homogeneous matches never benefit from positive losing prizes. This counterfactual analysis is also relevant to a more general job promotion setting with fixed prize budgets and similar skill–valuation profiles.

The remaining of the paper is structured as follows. Section 1.2 inspects the related literature and stresses the contribution of this work. Section 1.3 characterizes the model and states main theoretical results. Section 1.4 describes the data and the structural setup, provides estimation results and counterfactual experiments. Section 1.5 concludes.

1.2 Related Literature

The paper contributes to several research fields. First, it is closely related to the contest design literature that focuses on the optimal prize allocation in different environments.[5] Second, the current paper also complements the empirical work testing the contest theory predictions. To the best of my knowledge, both fields never looked at the setting with multi-dimensional prizes and heterogeneous preferences over them.

My theoretical contribution can be summarized as follows. Lazear and Rosen (1981) analyzed a tournament with two players and one-dimensional rewards. They showed that managers who want to maximize aggregate effort must always assign the highest feasible difference between winning and losing prizes. On the contrary, this paper shows when large prize spreads in all reward dimensions lower the total effort. Barut and Kovenock (1998), Moldovanu and Sela (2001) analyzed contests with more than two ex-ante symmetric participants and a single perfectly divisible good. Both papers found it optimal to assign up to $N-1$ positive prizes for N contestants if the cost function is convex; otherwise, a winner must take all. This implicitly says that the designer who wants to induce the highest total effort in the single-good setting must never reward the weakest performer. In my model with two-dimensional prizes and ex-ante asymmetric contestants, higher losing benefits increase expected aggregate effort if heterogeneity in preferences and / or skills is severe. Importantly, this result does not rely on convexity of the effort cost function.

The models that investigate the effect of monetary and status incentives in contests show some similarities with the proposed multi-dimensional setting. Moldovanu et al. (2007) introduce a framework where ex-ante symmetric agents engage in a contest game, and the designer allocates

[5]Sisak (2008) provides a profound overview of the existing literature.

money or status in order to maximize expected aggregate effort. Importantly, the two prize dimensions are perfectly correlated. The authors show when it is optimal to implement a coarse partition into status groups, i.e. lump weaker performers together and give them the same losing prize. On the contrary, the top category must always include a single element to provide enough incentives to exert effort.

Dubey and Geanakoplos (2005) analyze contests where students with heterogeneous abilities compete for the prizes consisting of exam grades and the status derived from them. In this setting, coarse partitions result in more aggregate effort than grading on a curve because the former policy allows weaker participants to achieve top-ranks with a higher probability. The mechanism described in the current paper also incentivizes disadvantaged contestants to compete more. However, my setup, where no restriction on correlation is imposed, gives much more flexibility in the design-related aspects. For example, the schedules where a winner gets only one good and a loser takes another item can be optimal. This could not be the case in the aforementioned models with strongly correlated reward dimensions.

The paper also relates to the contest literature that focuses on the equilibrium characterization in the presence of consolation prizes. Baye et al. (2012) analyze symmetric Nash equilibria in all-pay auctions with two homogeneous players and different types of externalities. The latter feature allows a loser to obtain a positive payoff, which depends on the effort of his opponent, and makes a set of symmetric equilibria much larger than in a standard setting. In a very similar framework applied to a Tullock-type contest, Chowdhury and Sheremeta (2011) solve for a unique symmetric equilibrium. Both papers, however, do not address any design-related issues and take the prize allocation as given. On the contrary, the present work proves when positive losing benefits become optimal from the contest organizer's prospective. Importantly, this result does not rely on externalities, but requires a sufficient degree of heterogeneity and multiple prize items.

Further, I highlight the contribution to the empirical literature on contests. Following standard theoretical models, this field does not address multi-dimensionality issues and consider only single-item rewards. The experimental evidence on prize incentive effects in contests is mixed. Most of the laboratory tests support the prediction that higher prize spreads induce more aggregate effort.[6] At the same time, field experiments find conflicting evidence. For example, De Paola et al. (2016) introduce some elements of bilateral rank-order contests to the high-school grading system. They observe that growing prize spreads have no or a negative effect on students' performance, which goes against the standard contest theory. However, the mechanism I describe in the current paper could explain this finding as follows. If students have multi-dimensional preferences (for example, they also care about status), but the designer is not aware of this, higher prize spreads in grades can reduce effort in relatively heterogeneous matches.

[6]See Dechenaux, Kovenock, and Sheremeta (2015) for a review.

Non-experimental tests of the contest theory mainly deal with professional sports data. This setup is very close to the job promotion setting, but it has a clear advantage in terms of available proxies for contestants' heterogeneity and effort choices. The vast majority of non-experimental tests use the reduced-form approach and follow the standard contest theory by looking at single-item prizes. Ehrenberg and Bagnano (1990), Maloney and McCormick (2000), Sunde (2003), Azmat and Moller (2008) test incentive effects of monetary rewards in running competitions and at the final stages of professional tennis tournaments. They find that higher monetary prize spreads increase effort and improve contestants' performance, which is in line with the standard theoretical predictions. Sunde (2009), Brown (2011), Berger and Nicken (2014) confirm that professional athletes tend to exert less effort when the contest becomes more heterogeneous.

The current paper also employs data from professional sports, namely tennis. In contrast to the aforementioned reduced-form tests, I introduce two reward dimensions (i.e. money and the ATP ranking points) and structurally recover contestants' skills and preferences. Although this estimation approach is neglected in the empirical contest literature, it has obvious advantages when one must restore utility parameters and run policy experiments.[7] Importantly, overlooking the non-monetary dimension results in overestimated incentive effects of monetary prizes and underestimated heterogeneity. Similar to the reduced-form tests, I provide evidence that contestants tend to exert less effort in more uneven matches. Higher prize spreads, however, reduce expected aggregate effort in relatively heterogeneous competitions, and this contrasts with the previous empirical tests of the contest theory.

1.3 The Model

1.3.1 Model Setup

I analyze a complete information contest with two heterogeneous participants.[8] This setting characterizes many interactions. Often job promotion contests for top positions or lobbying games do not have more than two participants. In professional sports such as tennis, football, basketball etc. only two players or teams compete in a particular tournament round. Other reasons I restrict the model in this way are tractability and exposition clarity.

[7] Structural estimation is widely used in the applied auction and industrial organization literature. To the best of my knowledge, however, this is the first empirical test of the contest theory that structurally estimates the incentive effects of prize schedules.

[8] In this section, I do not introduce private information to keep the analysis as simple as possible but relax this assumption later on. Moreover, in many contest-type settings participants know each other's characteristics. Often workers willing to be promoted are colleagues who have enough information about the opponent. In professional tennis, players observe competitors rankings, have some information about their training conditions, injuries etc.

The designer has two non-tradable perfectly divisible goods, A and B, and there is no market to exchange the items.[9] Without loss of generality, I assume equal endowments of goods and normalize them to a unity: $\hat{A} = \hat{B} = 1$. Let $W = \begin{pmatrix} A^W \\ B^W \end{pmatrix}$ and $L = \begin{pmatrix} A^L \\ B^L \end{pmatrix}$ be winning and losing bundles, respectively.

Contestant i attaches valuation α_i (β_i) to good A (B). Players' preferences are characterized by risk-neutrality and perfect additive separability:

$$U_i\left(A^k,\, B^k\right) = \alpha_i A^k + \beta_i B^k \equiv U_i^k,\; i = \{1,\, 2\}\,,\; k = \{W,\, L\}$$

Competitors choose non-negative effort, e_i, simultaneously and pay a cost $\gamma_i\left(e_i\right) = e_i$.[10] Player i wins if he exerts more effort than the opponent, and ties are broken randomly. Then, if contestant i obtains prize k, his payoff looks as follows:

$$\pi_i\left(A^k,\, B^k,\, e_i\right) = U_i\left(A^k,\, B^k\right) - e_i \equiv \pi_i^k$$

Given the reward schedule and the competitor's strategy, player i chooses his effort in order to maximize the expected payoff:

$$\max_{e_i} \left\{P_{\{e_i > e_{-i}\}} U_i\left(A^W,\, B^W\right) + \left(1 - P_{\{e_i > e_{-i}\}}\right) U_i\left(A^L,\, B^L\right) - e_i\right\}$$

where $P_{\{e_i > e_{-i}\}} \in [0,\, 1]$ is a probability that player i wins. Here I allow for both pure and mixed strategy equilibria in the game between contestants.

The designer observes only players' relative standing and chooses the prize allocation that maximizes contestants' aggregate effort. The game proceeds as follows:

1. The designer assigns W and L and commits to this reward schedule.

2. Contestants select e_i, $i = \{1,\, 2\}$ taking the prize allocation as given.

1.3.2 Contestants' Equilibrium Behavior

Further, I analyzing contestants' equilibrium behavior. Let $\triangle_A = A^W - A^L$ and $\triangle_B = B^W - B^L$ be prize spreads in corresponding reward dimensions. To avoid confusions with the case of heterogeneous abilities, I do not label players as "strong" and "weak". Instead, contestants' types are defined as follows:

Definition 1.1. *Let $t_i\left(\triangle_A,\, \triangle_B\right) = U_i^W - U_i^L \equiv t_i$ be contestant i's winning benefit, and define $t_g = \max\left\{t_1,\, t_2\right\}$, $t_n = \min\left\{t_1,\, t_2\right\}$. Then, i is the "greediest-to-win" ($g = i$) if and only if $t_i = t_g$; otherwise, i is not the "greediest-to-win" ($n = i$).*

[9] If this is not the case, one goes back to a standard contest model with one-dimensional prizes.

[10] For now, I impose no heterogeneity in contestants' abilities but relax this assumption later on.

With this formulation, players' types depend on the prize allocation. Then, the optimization problem of contestant i can be rewritten in terms of his type:

$$\max_{e_i} \left\{ P_{\{e_i > e_{-i}\}} t_i + U_i^L - e_i \right\}$$

Since t_g and t_n are functions of contestants' valuations and the reward schedule, types can be either positive or negative, and this will affect a structure of the equilibrium played.

Definition 1.2. *The equilibrium is trivial if and only if at least one player chooses $e = 0$ with probability 1. Otherwise, the equilibrium is non-trivial.*

Proposition 1.1 shows when there exists a non-trivial equilibrium and characterizes contestants' strategies for different realizations of t_n:

Proposition 1.1. *For $t_n \leq 0$ the equilibrium is always trivial. For $t_n > 0$ the equilibrium is unique and non-trivial:*

- *Contestant g randomizes uniformly on $[0,\, t_n]$, and his equilibrium payoff is $\pi_g = t_g - t_n + U_g^L \geq U_g^L$.*
- *Contestant n randomizes uniformly on $(0,\, t_n]$, places the atom of size $p_n^0 = \frac{t_g - t_n}{t_g}$ at $e = 0$, and his equilibrium payoff is $\pi_n = U_n^L \leq \pi_g$.*

Proof. See Appendix E. □

The non-trivial equilibrium from Proposition 1.1 is similar to the one Baye et al. (1996) derive for asymmetric all-pay auctions with two bidders and complete information. However, their characterization does not extend to the case of perfectly divisible goods and heterogeneous losing benefits.[11] Also, Baye et al. (1996) cannot accommodate prize-dependent types. Hence, Proposition 1.1 provides a more general characterization than the aforementioned work.[12]

As Proposition 1.1 states, two equilibrium configurations can emerge. When there is no strictly positive winning benefit for type n ($t_n \leq 0$), this player always prefers to stay inactive, i.e. $e_n = 0$. If type n deviates towards $e_n > 0$ and wins, his payoff becomes $\pi_n(e_n) = t_n + U_n^L - e_n$. However, $\pi_n(e_n)$ is strictly dominated by $\pi_n(0) = U_n^L$ for any $e_n > 0$. This equilibrium corresponds to the worst possible scenario from the designer's prospective. On one hand, type n never exerts positive effort. On the other hand, the lack of competition makes winning easier for player g and drives e_g down. As a result, the trivial equilibrium leads to the lowest aggregate effort possible.

[11] In a single-object auction setting, players with lower bids always get nothing

[12] The Baye et al. (1996) case is nested in the current equilibrium characterization. If one takes a one-dimensional reward schedule and fixes a winning prize at a unity, he gets exactly the same setup as Baye et al. (1996).

On the contrary, if type n has incentives to compete ($t_n > 0$), both players choose $e > 0$ with a strictly positive probability. Then, contestants use mixed strategies, and ties happen with a zero probability. Importantly, type n can never get higher equilibrium payoff than his advantaged competitor. Since player n does not exert effort that exceeds his type, the greediest participant can always choose $e_g \in [t_n, t_g)$ and win with certainty.[13] Contestants' equilibrium payoffs are equal if and only if the types coincide ($t_n = t_g$) and no positive losing prizes are assigned.

One could also interpret the atom contestant n places at zero (p_n^0) as a relative power of player g. When $p_n^0 \to 1$, the greediest type is extremely strong, and this destroys his opponent's incentives to compete. If type g has zero relative power, i.e. $p_n^0 = 0$, the contest is equivalent to the homogeneous one.

Using Proposition 1.1, I can write down contestants' expected effort in a closed form :

$$E_g = \frac{t_n}{2},\ E_n = \left(1 - p_n^0\right)\frac{t_n}{2}$$

To emphasize the difference between one- and two-dimensional prizes, assume good A is not available. In this case, type g's relative power looks as follows:

$$p_n^0 = \frac{\beta_g - \beta_n}{\beta_g}\,\forall \triangle_B$$

and this value is constant. In other words, no matter how high stakes are, type n stays inactive with the same probability. Thus, the only element of the expected effort that depends on $\triangle_B$ is t_n, the winning benefit of contestant n. Since $\frac{\partial t_n}{\partial \triangle_B} = \beta_n > 0$, lower prize spreads decrease expected effort of both contestants. Hence, one can already anticipate that with one-dimensional rewards the designer will always implement the "Winner–Takes–All" allocation. This result is well-known in the contest design literature.

Now suppose both dimensions become available. Then, the relative power of player g depends on prize spreads, $\triangle_A$ and $\triangle_B$:

$$\frac{\partial p_n^0}{\partial \triangle_A} = -\left(\alpha_n\beta_g - \alpha_g\beta_n\right)\frac{\triangle_B}{t_g^2}$$
$$\frac{\partial p_n^0}{\partial \triangle_B} = \left(\alpha_n\beta_g - \alpha_g\beta_n\right)\frac{\triangle_A}{t_g^2}$$

When contestants have unequal marginal rates of substitution ($\frac{\alpha_n}{\beta_n} \neq \frac{\alpha_g}{\beta_g}$), these derivatives are typically of different signs.[14] For example, take the "Winner–Takes–All" prize allocation ($\triangle_A = \triangle_B = 1$). If contestant g is more sensitive to incentives in dimension A than the opponent, his relative power must decrease (increase) in $\triangle_B$ ($\triangle_A$) :

$$\triangle_A = 1,\ \frac{\alpha_n}{\beta_n} > \frac{\alpha_g}{\beta_g} \Rightarrow \frac{\partial p_n^0}{\partial \triangle_A} > 0,\ \frac{\partial p_n^0}{\partial \triangle_B} < 0$$

[13] If type n chooses $e_n = t_n + \varepsilon$, $\varepsilon > 0$ and wins, he gets $\pi_n(e_n) = U_n^L - \varepsilon$, and this is dominated by $e_n = t_n$.

[14] If the latter condition is violated, type g's relative power does not respond to changes in the prize allocation.

Next, suppose the designer reduces the prize spread in dimension A. First of all, this policy drives winning benefits of both players (t_g and t_n) down, and they have less incentives to exert effort (direct effect). On the other hand, lower prize spreads in dimension A undermine type g's relative power and make the contest more even. As a result, player n is incentivized to exert higher effort (equilibrium effect). When the latter positive effect dominates the negative one induced by the reduction of winning benefits, expected aggregate effort raises.[15] I derive necessary and sufficient conditions for this to hold when characterize the optimal prize allocation.

1.3.3 The Optimal Prize Allocation

The designer chooses the prize allocation that maximizes expected aggregate effort and does not violate feasibility constraints:

$$J = \max_{A^W,\,A^L,\,B^W,\,B^L} \left[E_g + E_n\right]$$
$$s.t.\, A^k,\, B^k \geq 0,\, k = \{W,\, L\}$$
$$A^W + A^L \leq 1,\, B^W + B^L \leq 1$$

where $E_g = \frac{\alpha_n \triangle_A + \beta_n \triangle_B}{2}$ and $E_n = \left(\frac{\alpha_n \triangle_A + \beta_n \triangle_B}{\alpha_g \triangle_A + \beta_g \triangle_B}\right) \frac{\alpha_n \triangle_A + \beta_n \triangle_B}{2}$ are contestants' expected effort.

1.3.3.1 Fixed Prize Allocation in One Dimension

I start from the case when prizes in one dimension are fixed, i.e. the designer optimizes only over a particular good. Often contest managers act under similar constraints. Executives in organizations can decide about monetary rewards but must take the hierarchical structure of the company (workers' status concerns) as given. In professional tennis, where money and ranking points are distributed, managers have no power to change prizes in the latter dimension because they are fixed by the ATP. Also, this restricted case is easier to analyze, and I use it as a building block for the main theoretical result later on.

Without loss of generality, assume that prizes in dimension B are fixed:

$$B^W = \hat{B}^W,\, B^L = \hat{B}^L \Rightarrow \hat{\triangle}_B = \hat{B}^W - \hat{B}^L$$

and the designer optimizes only over dimension A including a possibility to leave good B out. Let $r = \{\alpha_g,\, \alpha_n,\, \beta_g,\, \beta_n\}$ be contestants' valuation profile, and R denotes a set of all feasible r's:

$$R = \{r :\, \alpha_i \geq 0,\, \beta_i \geq 0,\, i = \{g,\, n\}\}$$

[15]Formally, E_g has only one term, t_n, depending positively on the prize spreads. Hence, type g's effort never grows when $\triangle_A$ becomes lower. However, player n's expected effort includes two elements, $\left(1 - p_n^0\right)$ and t_n, that change in opposite directions when $\triangle_A$ decreases. If growth in $\left(1 - p_n^0\right)$ over-compensates losses in expected effort caused by lower winning benefits, E_n increases.

The optimal prize allocation depends on contestants' preferences. Proposition 1.2 characterizes all possible choices the designer can make given the valuation profile:

Proposition 1.2. *For any $\hat{\triangle}_B \in [-1,\, 1]$ and valuation profile $r \in R$ the designer*

- *Uses both goods and leaves a positive losing prize in dimension A, or*
- *Uses both goods and gives the endowment of A to a winner, or*
- *Does not use dimension B and gives the endowment of A to a winner.*

Proof. See Appendix E.[16] □

In the proof, I study properties of the designer's objective function, $J(\cdot)$, that can be expressed in terms of $\triangle_A$ and $\hat{\triangle}_B$.[17] First, $J(\cdot)$ is strictly convex in $\triangle_A$, and the derivative of $J(\cdot)$ with respect to $\triangle_A$ is discontinuous at $\tilde{\triangle}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n}\hat{\triangle}_B$.[18] When the prize spread in dimension A is equal to $\tilde{\triangle}_A$, both contestants get identical winning benefits (namely, $t_g = t_n$). Let $\bar{\triangle}^i_A$, $i = \{1,\, 2\}$ denote the allocation of item A that results in zero winning benefits for player i:

$$\bar{\triangle}^i_A = \left\{\triangle_A :\, \alpha_i \triangle_A + \beta_i \hat{\triangle}_B = 0\right\}$$

Choosing $\triangle_A \leq \max\left\{\bar{\triangle}^1_A,\, \bar{\triangle}^2_A\right\}$ ($\triangle_A > \max\left\{\bar{\triangle}^1_A,\, \bar{\triangle}^2_A\right\}$), the designer induces zero (positive) expected aggregate effort (see Proposition 1.1). If $\tilde{\triangle}_A \leq \max\left\{\bar{\triangle}^1_A,\, \bar{\triangle}^2_A\right\}$, the objective function $J(\cdot)$ is strictly increasing in $\triangle_A \in \left[\max\left\{\bar{\triangle}^1_A,\, \bar{\triangle}^2_A\right\},\, 1\right]$. In this case, the designer must give the endowment of good A to a winner. Under $\tilde{\triangle}_A > \max\left\{\bar{\triangle}^1_A,\, \bar{\triangle}^2_A\right\}$, expected aggregate effort is strictly increasing in $\triangle_A$ for any $\triangle_A \in \left[\max\left\{\bar{\triangle}^1_A,\, \bar{\triangle}^2_A\right\},\, \tilde{\triangle}_A\right)$, but can be non-monotone on the interval $\triangle_A \in \left[\tilde{\triangle}_A,\, 1\right]$. Given that $J(\cdot)$ is strictly convex, there is no interior maximum in $\triangle_A \in \left(\tilde{\triangle}_A,\, 1\right)$. Thus, the two candidate prize bundles: $\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right)$ and $\left(1,\, \hat{\triangle}_B\right)$ – must be compared directly. In the proof of Proposition 1.2, I characterize valuation profiles for which $\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right)$ is preferred to $\left(1,\, \hat{\triangle}_B\right)$, i.e. when giving a positive prize to a loser improves expected aggregate effort. Finally, since the designer cannot change the allocation of item B but is allowed to leave it out, he must directly compare expected aggregate effort induced by the best bundle and its counterpart in a single-good contest over dimension A. Importantly, there exist valuation profiles such that the designer prefers the latter option.

[16] When rewards in dimension A are fixed, the sets of valuation profiles and the optimal prize allocation look exactly the same if one replaces α with β and A with B in the proof.

[17] When prizes are fully flexible, I use Proposition 1.2 to find the best reward schedule in dimension A given the optimal allocation of item B.

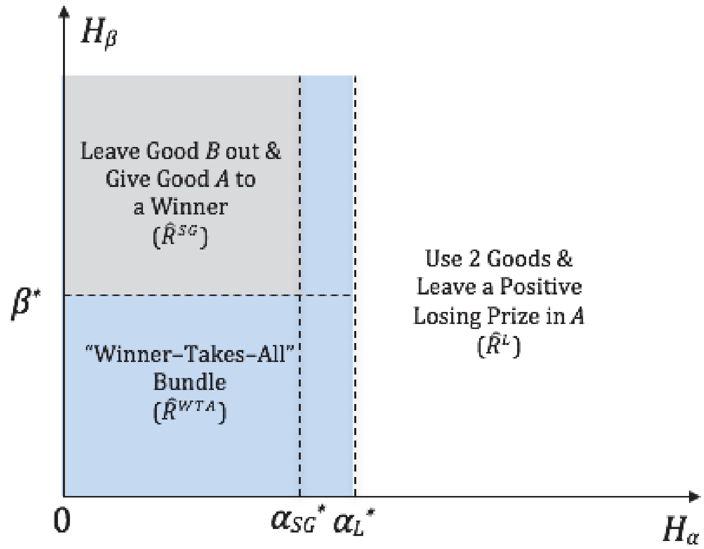
[18] The designer's objective function has a kink at $\tilde{\triangle}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n}\hat{\triangle}_B$, and this point can be either feasible or infeasible.

Proposition 1.2 partitions a set of valuation profiles (R) into three subsets:

$$R = \hat{R}^L \cup \hat{R}^{WTA} \cup \hat{R}^{SG}$$

where $\hat{R}^L$ ($\hat{R}^{WTA}$) corresponds to optimal bundles with positive (zero) losing prize, $\hat{R}^{SG}$ stands for contests with single-item rewards in dimension A. More precisely, the designer's choice is driven by the degree of heterogeneity in players' valuations. In the beginning, I summarize these findings with a simple diagram and then explain the intuition. Let $H_\alpha = \alpha_g - \alpha_n$ ($H_\beta = \beta_g - \beta_n$) be the degree of heterogeneity between contestants in dimension A (B). Also, assume that type g values both goods more than his competitor ($\alpha_g > \alpha_n$, $\beta_g > \beta_n$). Figure 1.3.1 shows how the designer's choice depends on H_α and H_β in case of $\hat{\triangle}_B \geq 0$. I keep α_g (β_g) constant and attribute all changes in H_α (H_β) to α_n (β_n).

Figure 1.3.1: Optimal Prize Allocation as a Function of Contestants' Heterogeneity: $\hat{\triangle}_B \geq 0$



Note: $H_\alpha = \alpha_g - \alpha_n$ ($H_\beta = \beta_g - \beta_n$) corresponds to the degree of heterogeneity between contestants in dimension A (B); α_g (β_g) is kept constant, $\alpha_g > \alpha_n$ and $\beta_g > \beta_n$; values with asterisks denote thresholds imposed on the degree of contestants' heterogeneity in different dimensions.

Although the prizes in good B are fixed, heterogeneity in both dimensions affects the designer's choice. The "Winner–Takes–All" allocation is optimal when contestants have similar preferences over reward items. If heterogeneity in dimension A is severe, the designer prefers to leave a positive losing prize in good A. Finally, when preferences over item A but not B are relatively homogeneous, neglecting the latter dimension results in the highest expected aggregate effort.

Optimal positive losing prizes in dimension A require enough asymmetry in contestants' valuations. Importantly, player g must be more sensitive to incentives shaped by good A than his competitor:

$$\begin{cases} \alpha_g > \alpha_n \\ \frac{\alpha_g}{\beta_g} > \frac{\alpha_n}{\beta_n} \end{cases}$$

A sufficient condition to make $A^L > 0$ optimal is strong heterogeneity in dimension A.

Since contestants' expected effort depends on the prize spreads, one can think about the equivalence of the following alternatives:

1. Use the endowment of good A completely and increase the losing prize at the cost of the winning reward ($\triangle_A = \tilde{\triangle}_A < 1$ and $A^W + A^L = 1$), or

2. Waste some endowment of good A, assign zero losing (winning) benefit in this item and give the winner (the loser) $A^W = \tilde{\triangle}_A < 1$ in case of $\tilde{\triangle}_A \geq 0$ ($A^L = -\tilde{\triangle}_A < 1$ in case of $\tilde{\triangle}_A \in (-1, 0)$).

The designer, however, prefers option 1 (no waste) and does not destroy the endowment of item A. Remarkably, when type g values both goods more and players' preferences in dimension A differ a lot, the designer benefits by assigning $\triangle_A < 0$, i.e. the prize spread becomes negative.[19] The latter observation makes the result robust to the introduction of costly prizes: even then the designer will be willing to reward the weakest performer in case of severe heterogeneity.

Overall, positive losing prizes affect the effort players exert in two ways:

1. Since winning benefits decrease, both contestants have less incentives to fight. I call this **direct effect.**

2. When player i reduces his effort, the opponent is willing to compete more and, consequently, wins with a higher probability. I label this indirect response to growing losing prizes as **equilibrium effect**.

Suppose $\tilde{\triangle}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n}\hat{\triangle}_B$ is feasible.[20] If the designer implements the $\left(\tilde{\triangle}_A, \hat{\triangle}_B\right)$ allocation, players' winning benefits (i.e. their types) look as follows:

$$t_g = t_n = \frac{\alpha_g\beta_n - \alpha_n\beta_g}{\alpha_g - \alpha_n}\hat{\triangle}_B \equiv \tilde{t}$$

and the contest becomes homogeneous. When type g prefers good A relatively more than his opponent ($\alpha_g > \alpha_n$ combined with $\frac{\alpha_g}{\beta_g} > \frac{\alpha_n}{\beta_n}$) and $\hat{\triangle}_B > 0$, the value of $\tilde{t}$ is positive. Importantly, the increase in losing benefits does not destroy contestants' incentives to compete. Moreover, the $\left(\tilde{\triangle}_A, \hat{\triangle}_B\right)$ allocation redistributes the relative power in favor of type n:

[19] In case of extreme heterogeneity in dimension A, the designer prefers to give the endowment of this good to a loser.

[20] When $\tilde{\triangle}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n}\hat{\triangle}_B$ is infeasible, the designer assigns the lowest possible prize spread in dimension A (i.e. $\triangle_A = -1$).

$$\tilde{p}_n^0 = 1 - \frac{\tilde{t}}{\tilde{\tilde{t}}} = 0$$

Then, player n is more likely to succeed, and his incentives to compete increase compared to the "Winner–Takes–All" schedule. Thus, higher losing prizes reinforce the positive equilibrium effect.

In the one-dimensional setting, the designer can never make the contest homogeneous and at the same time provide sufficient stimuli to exert effort. Assume good B is not available:

$$t_g = \alpha_g \triangle_A,\ t_n = \alpha_n \triangle_A$$

To equalize contestants' types, the designer must assign identical winning and losing prizes (i.e. $\triangle_A = 0$). This, however, destroys incentives completely, and in equilibrium, both participants exert zero effort. In the two-dimensional case, the designer not only makes the match even but also preserves strictly positive winning benefits. This mechanism still works when one introduces heterogeneity in skills and asymmetric information about players' types to the model.[21]

Another key ingredient that allows the designer to affect the relative power of type g is asymmetry in contestants' valuations (i.e. $\frac{\alpha_g}{\beta_g} \neq \frac{\alpha_n}{\beta_n}$). The winning benefit of player g must decline faster than the one of his opponent when losing prizes grow.[22] On the contrary, collinear preferences ($\frac{\alpha_g}{\beta_g} = \frac{\alpha_n}{\beta_n}$) will make the designer unable to redistribute the relative power without destroying incentives to compete (in this case, $\tilde{t} = 0$ holds). The latter setting has same qualitative properties as the one-dimensional model where the "Winner–Takes–All" schedule maximizes expected aggregate effort.

Now, I characterize how heterogeneity affects the optimal prize allocation (see Figure 1.3.2). When contestants have similar preferences, player g does not have a significant advantage over the opponent. Further, winning benefits of both types (namely, g and n) decline more or less symmetrically when losing rewards grow (see Figure 1.3.2). In this case, the positive equilibrium effect will never compensate losses associated with the winning prize reduction (direct effect). As a result, the designer implements the "Winner–Takes–All" allocation and does not redistribute the relative power between competitors if heterogeneity is weak.

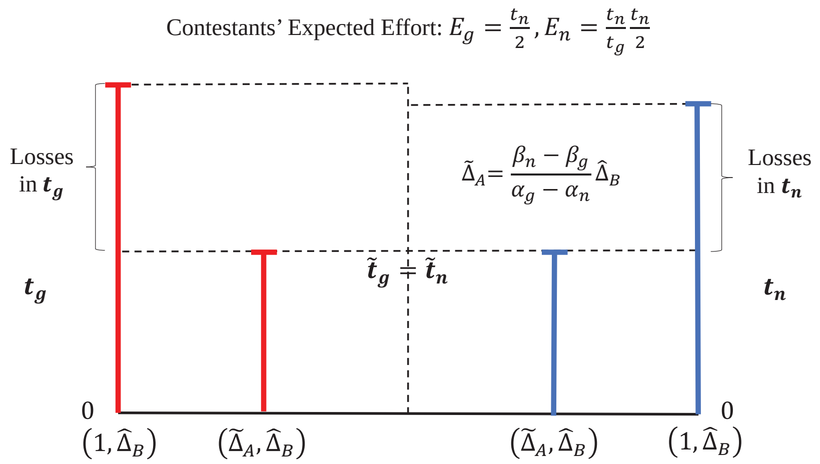
When the advantage of type g is strong, the opponent does not have enough incentives to compete. As a consequence, the "Winner–Takes–All" scheme cannot support the highest expected aggregate effort. On the contrary, positive losing prizes make players' types more balanced (ideally, $t_n = t_g$), and the contest becomes homogeneous. Since preferences are asymmetric, the policy reduces winning benefits of the advantaged competitor significantly but does not affect the opponent much (see Figure 1.3.2). Thus, the positive equilibrium effect dominates the direct one, and higher losing prizes increase expected aggregate effort.

[21] See Subsection 1.3.4 and Appendix A, respectively.

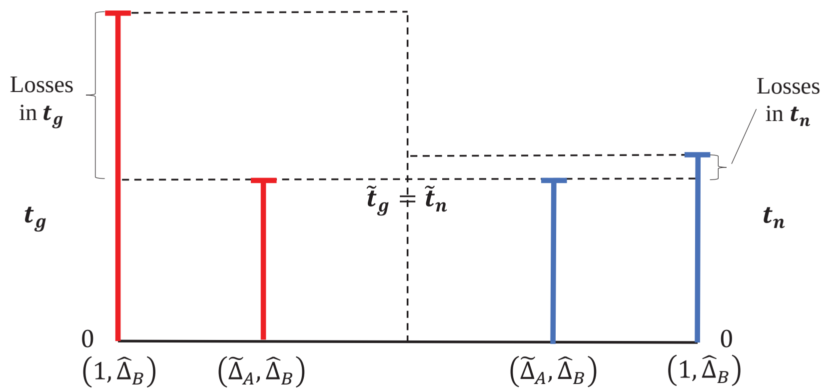
[22] Since the reward structure in dimension B is fixed, all changes in winning benefits are driven by shifts in $\triangle_A$.

Figure 1.3.2: Optimal Losing Prizes and Contestants' Heterogeneity

Weak Heterogeneity in Dimension A



Strong Heterogeneity in Dimension A



Finally, I comment on the preference profiles entering $\hat{R}^{SG}$. When the positive losing prize in item A is optimal (i.e. $\triangle_A = \tilde{\triangle}_A$), the designer never ignores dimension B. Given that the $\left(\tilde{\triangle}_A, \hat{\triangle}_B\right)$ allocation requires strong heterogeneity in preferences over good A, a single-item reward schedule in this dimension does not result in sufficient effort. However, with the second good at place, the designer can balance players' incentives to compete and make the contest homogeneous.

If preferences over item A are similar, the endowment of this good must be assigned to a winner ($\triangle_A = 1$). Given $\hat{\triangle}_B$ and contestants' valuations in dimension B, the designer must decide whether to use item B or leave it out. In particular, for $\hat{\triangle}_B \geq 0$ he prefers to run a single-good contest in dimension A under following conditions:

1. Players respond to incentives in dimension A the most ($\alpha_i \gg \beta_i$);

2. Contestant g values both goods more than the opponent ($\alpha_g \geq \alpha_n$, $\beta_g > \beta_n$);

3. Heterogeneity in dimension B (A) is sufficiently strong (weak), and contestant g is more sensitive to changes in B^W and B^L, i.e. $\frac{\beta_g}{\alpha_g} > \frac{\beta_n}{\alpha_n}$.

To highlight the intuition, take the case of relatively homogeneous preferences in dimension A ($\alpha_g \approx \alpha_n$). When the designer uses item B, winning benefits grow (direct effect). At the same time, the competition becomes more heterogeneous: since $\beta_g > \beta_n$, the advantage of type g rises. As a result, in equilibrium, player n has less incentives to exert effort (namely, the atom type n places at $e = 0$ increases). If heterogeneity in dimension B is strong, the gap in contestants' winning benefits increases significantly when item B is at place, and the negative equilibrium effect prevails. If the designer could adjust the prizes in good B, he would prefer to assign higher losing benefits and make the two-dimensional competition even. Since it is not feasible, the designer uses only good A and runs a homogeneous single-item contest. In this case, players still exert enough effort because they are mainly motivated by item A and heterogeneity in this dimension is weak.

If contestant g values only good A more than the opponent ($\alpha_g \geq \alpha_n$ combined with $\beta_g < \beta_n$), the bundle is always optimal. The presence of item B helps the designer to reduce the advantage of player g in dimension A and provide stronger incentives to compete (both in terms of direct and equilibrium effects).

1.3.3.2 Flexible Prize Allocation in Both Dimensions

In this subsection, I analyze the most general case when the designer can change the reward schedule in both dimensions. To characterize the optimal prize allocation, I use the results of Proposition 1.2. As before, R denotes a set of all feasible valuation profiles. Theorem 1.1 characterizes the designer's optimal choice given contestants' preferences:

Theorem 1.1. *For any valuation profile $r \in R$ the designer uses goods' endowments completely and either leaves a positive losing prize at least in one dimension or gives both items to a winner.*

Proof. See Appendix E. □

Theorem 1.1 employs the proof of Proposition 1.2 as a building block. I define the designer's objective function, $J(\cdot)$, in terms of $\triangle_A$ and $\triangle_B$. Now, $J(\cdot)$ has kinks at $\tilde{\triangle}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n} \triangle_B$ and

$\tilde{\triangle}_B = \frac{\alpha_n - \alpha_g}{\beta_g - \beta_n} \triangle_A$. Also, $J(\cdot)$ is strictly convex, and the designer's problem has no interior solution. Then, fix the prize allocation in dimension B, i.e. $\triangle_B = \hat{\triangle}_B$. Proposition 1.2 implies that for any $\hat{\triangle}_B$ there exists a non-empty set of valuation profiles such that the designer prefers to assign a positive losing prize in item A. Then, the statement must hold for $\hat{\triangle}_B = 1$ as well, and for some valuation profiles the "Winner–Takes–All" allocation is not optimal.

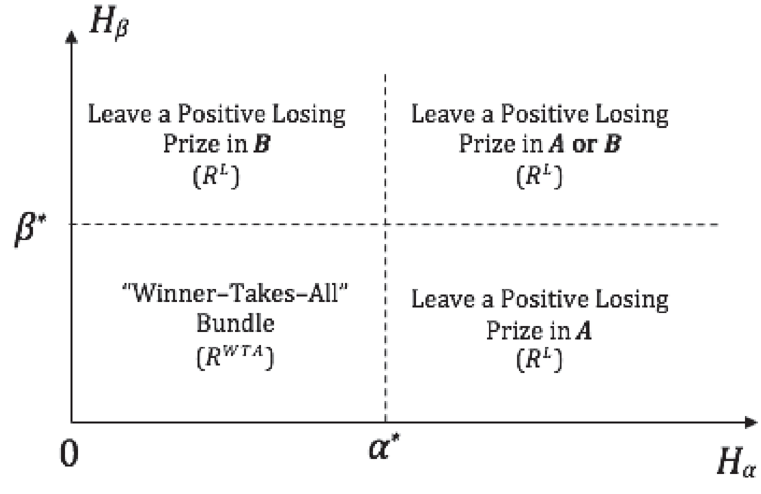
Using the result of Theorem 1.1, one can partition a set of valuation profiles (R) into two subsets:

$$R = R^L \cup R^{WTA}$$

where R^L (R^{WTA}) corresponds to bundles with positive (zero) losing prizes. Overall, properties of R^L and R^{WTA} are very similar to those of $\hat{R}^L$ and $\hat{R}^{WTA}$ from Proposition 1.2. However, with the fully flexible reward schedule, there do not exist valuation profiles such that it is optimal to ignore one good (recall $\hat{R}^{SG}$ from Proposition 1.2). This fact is not surprising given that the corresponding result in Proposition 1.2 was driven by impossibility to adjust the rewards in dimension B.

Figure 1.3.3 depicts the optimal prize allocation as a function of players' heterogeneity. When the contest is relatively even, the "Winner–Takes–All" bundle induces the highest expected aggregate effort. If players show strong heterogeneity in the dimension where type g's incentives are more sensitive (good A for $\frac{\alpha_g}{\beta_g} > \frac{\alpha_n}{\beta_n}$ and B, otherwise), it is optimal to assign a positive losing prize in the corresponding item.[23]

Figure 1.3.3: Optimal Prize Allocation as a Function of Contestants' Heterogeneity: Flexible Rewards in Two Dimensions



Note: $H_\alpha = \alpha_g - \alpha_n$ ($H_\beta = \beta_g - \beta_n$) corresponds to the degree of heterogeneity between contestants in dimension A (B); α_g (β_g) is kept constant, $\alpha_g > \alpha_n$ and $\beta_g > \beta_n$; values with asterisks denote thresholds imposed on the degree of contestants' heterogeneity in different dimensions.

[23]There also exist valuation structures such that it is optimal to give positive losing prizes in both items. Then the lowest prize spread must correspond to the dimension in which type g's incentives are more sensitive.

The mechanism driving the results was described in Subsection 1.3.3.1. On one hand, the positive losing prize reduces winning benefits and, consequently, incentives to compete (direct effect). At the same time, it destroys the relative advantage of the "greediest" player (g) that motivates the opponent to exert more effort (equilibrium effect). When contestants' preferences are very heterogeneous, the latter effect dominates, and expected aggregate effort increases. To illustrate all the points made, I provide a simple numerical example:

Example. Suppose $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$. The "Winner–Takes–All" schedule results in $J\,(1,\,0,\,1,\,0) \approx 14$. Player 1 is the "greediest-to-win" ($g = 1$) because his winning benefit exceeds the one of the opponent. For given α_g and α_n, it is always optimal to assign the highest prize spread in dimension A (see the proof of Theorem 1.1). Contestant g is more sensitive to incentives in dimension B ($\frac{\alpha_g}{\beta_g} < \frac{\alpha_n}{\beta_n}$). Hence, optimal positive losing prizes may appear only in the second good.

Now assume the designer implements another reward schedule in dimension B: $B^W = \frac{1}{4}$, $B^L = \frac{3}{4}$, and $\triangle_B = \tilde{\triangle}_B = -\frac{1}{2}$. This results in $J(1,\,0,\,\frac{1}{4},\,\frac{3}{4}) = 6$, and the "Winner–Takes–All" scheme dominates the latter allocation. In this case, the positive equilibrium effect is relatively small because contestants do not show enough heterogeneity in preferences over good B.

Next, take the same $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ combined with $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$. The "Winner–Takes–All" allocation results in $J\,(1,\,0,\,1,\,0) \approx 7.8$. As before, player 1 is the "greediest-to-win", and positive losing prizes may appear only in dimension B. The alternative reward scheme with $B^W = \frac{3}{7}$, $B^L = \frac{4}{7}$, and $\triangle_B = \tilde{\triangle}_B = -\frac{1}{7}$ induces $J(1,\,0,\,\frac{3}{7},\,\frac{4}{7}) \approx 9$. As a result, the designer benefits from the positive losing prize in dimension B. This is the case because contestants' preferences over good B differ a lot, and the equilibrium effect prevails.

1.3.4 Heterogeneity in Skills

Before I assumed that contestants have identical effort costs, and the only source of heterogeneity stems from preferences. However, in most empirical applications, players differ in abilities, experience, and other characteristics that can shape their skills. To account for this fact, I introduce player-specific cost functions. This setup is used in Section 1.4 to estimate contestants' skills and preferences.

Reformulate the model slightly. Assume contestant i has the following cost function:

$$\gamma_i\,(e) = \frac{e}{c_i},\; c_i > 0$$

where c_i is a skill parameter, $i = \{1,\,2\}$. Higher c_i makes effort exertion less costly. The expected payoff of player i looks as follows:

$$P\left(e_i > e_j\right) U_i^W + \left(1 - P\left(e_i > e_j\right)\right) U_i^L - \frac{e}{c_i}$$

Applying a monotone transformation, one can rewrite the problem contestant i solves:

$$\max_{e_i} \left\{ c_i \left[P\left(e_i > e_j\right) U_i^W + \left(1 - P\left(e_i > e_j\right)\right) U_i^L \right] - e_i \right\},\, i \neq j$$

Now, winning benefits of both players depend on valuations and skills. Define $\tilde{U}_i^W = c_i U_i^W$, $\tilde{U}_i^L = c_i U_i^L$, and $\tilde{t}_i = c_i\left(U_i^W - U_i^L\right)$. Replacing t_i with $\tilde{t}_i$ brings us back to the original model. The equilibrium characterization can be reformulated in terms of skill-valuation profiles, $\tilde{r} = \left\{\tilde{\alpha}_g,\, \tilde{\alpha}_n,\, \tilde{\beta}_g,\, \tilde{\beta}_n\right\}$, where $\tilde{\alpha}_i = c_i \alpha_i$, $\tilde{\beta}_i = c_i \beta_i$. Let $\tilde{R}$ be a set of all feasible $\tilde{r}$'s:

$$\tilde{R} = \left\{\tilde{r} :\, \tilde{\alpha}_i \geq 0,\, \tilde{\beta}_i \geq 0,\, i = \{g,\, n\}\right\}$$

Proposition 1.3 characterizes the optimal prize allocation for given skill-valuation profiles:

Proposition 1.3. *For any skill-valuation profile $\tilde{r} \in \tilde{R}$ the designer uses goods' endowments completely and either leaves a positive losing prize at least in one dimension or gives both items to a winner.*

Proof. The new optimization problem is a rescaled version of the original program. The proof follows immediately when one replaces the elements of R with those of $\tilde{R}$ in Theorem 1.1. □

Proposition 1.3 mirrors Theorem 1.1 applied to the case of heterogeneous skills. As before, optimal positive losing prizes require asymmetry in contestants' valuations and strong heterogeneity in the dimension preferred by player g relatively more. The mechanism driving this result was described in Subsection 1.3.3.1.

Skills can contribute to players' heterogeneity in different ways. Let H^A ($\tilde{H}^A$) be a degree of heterogeneity over dimension A in the original model (the setup with different skills):

$$H^A = \alpha_g - \alpha_n,\, \tilde{H}^A = c_g \alpha_g - c_n \alpha_n$$

Depending on c_g and c_n, heterogeneous skills can mitigate or amplify asymmetries in contestants' preferences.

This relatively simple model with no private information already helps in explaining a co-existence of positive winning and losing prizes. The results derived above could never be obtained in a standard single-item framework. Remarkably, optimality of non-zero losing benefits does not rely on convexity of the cost function as a traditional argument in favor of multiple positive prizes.[24] However, even then it would never be optimal to reward the weakest performer in case of single-item rewards.

In Appendix A, I extend the original model in two ways. First, I allow the designer to run separate competitions over both reward dimensions instead of making the prize bundle. With this

[24]For instance, see Moldovanu and Sela (2001), Barut and Kovenock (1998).

setting, players exert effort in contests A and B and get single-item rewards in corresponding items. Since prizes are costless for the designer, he can always unbundle goods unless specific constraints are imposed. If aggregate effort taken over two separate contests always dominates the one induce by bundling schemes, the results of Theorem 1.1 cannot be generalized and break down easily.

Theorem 1.1 shows that there exist preference profiles for which bundles (including those with positive losing prizes) generate strictly more expected aggregate effort than two simultaneous single-item contests. The intuition is as follows. Bundles with positive losing prizes become optimal when players' preferences display strong heterogeneity. This also means that corresponding single-item contests are very uneven and result in low expected aggregate effort. Hence, the two-dimensional allocation with positive losing benefits helps the designer to mitigate a negative effect heterogeneity has on players' incentives to exert effort.

Another extension introduces asymmetric information about contestants' types. This modification is important for two reasons. First, players can have unobservable characteristics that affect their effort choices. Second, often the designer must commit to the prize allocation before he learns the exact matching. For example, tournament organizers in professional tennis announce monetary reward schedules before the final draw is known.

Suppose that contestants' preferences over dimension A (α_i) constitute their private information. For simplicity, I assume that α_i has only two realizations: $\alpha_i = \{\underline{\alpha}_i, \bar{\alpha}_i\}$ with $P(\alpha_i = \bar{\alpha}_i) = k$ – and α_1, α_2 are drawn independently. I find that there exist probability-valuation profiles, defined over k and contestants' preferences, for which it is optimal to make a bundle with a positive losing prize. The underlying intuition is the same as described in Subsection 1.3.3.1.

1.4 The Empirical Application

This section estimates the model and introduces policy experiments. The most important application of the proposed theoretical framework is job promotion with monetary and non-monetary rewards. Unfortunately, there do not exist sufficient firm-level data that approximate contestants' heterogeneity and their effort choices. Nevertheless, this information can be restored if one analyzes professional sport competitions. The setup does not differ much from the job promotion context. Athletes often have two-dimensional incentives shaped by monetary prizes and career concerns (their position in the ranking). At the same time, they show enough heterogeneity in skills. Thus, the results obtained with the sports data generalize to the job promotion and other similar settings.

To estimate the model, I use a sample of first-round games from two highly prestigious tennis tournaments, the Australian Open and the French Open. These contests display the key features of the theoretical model. Importantly, players care about two goods: prize money and their position in the ranking. Regarding the latter dimension, each year tennis players participate in a sequence of tournaments and earn points. The rankings for males and females are updated and released every week.

Obviously, the indicated prize dimensions display positive correlation. However, there is no market to exchange the two goods of interest. One can argue the ATP ranking points matter for contestants' effort exertion. In fact, this prize dimension maps into career concerns. Being the top player gives an easier access to other tournaments, media coverage, recognition, contracts for participation in advertising campaigns etc. Further, in the end of every game season top–8 male athletes are nominated to play in the ATP World Tour Finals, an extremely prestigious competition.[25] Another argument stressing the relevance of the ranking is the existence of well-recognized tournaments with no money at stake. For instance, in 2009–2015 the Davis Cup allocated only the ATP points.[26] Nevertheless, top–athletes such as Novak Djokovic, Andy Murray, Stan Wawrinka participated in the competition. Later on, I provide more evidence that the ATP ranking points matter for contestants' effort exertion (see Subsection 1.4.2.2).

Tennis players show enough variation in skills. Moreover, their preferences over prize dimensions can differ too. Given the empirical distribution of ranking points (see Figure 1.5.1), 10 additional units of the good can never change positions of top–10 players. However, the same amount may move athletes ranked 90–100 up to 9 positions (keeping points of their closest peers constant). Finally, tennis players are professionals who perfectly know the rules of the game and act strategically.[27] Thus, one should expect them to behave rationally in the given environment.

Importantly, contest organizers in professional tennis cannot change the allocation of ranking points between tournament stages. This element of the prize schedule is fixed by the ATP and the WTA for males and females, respectively. However, tournament managers have a power to vary monetary rewards. Very often, they leave positive prizes for first-round losers. For example, in 2013, the Australian Open increased this element of the reward schedule significantly. At the same time, other prizes did not evolve as much (see Tables 1.6 and 1.7). Both contest managers and the players expected this policy to induce more competition at earlier stages of the tournament.[28] However, one could come with alternative explanations of the changes observed. The first argument in favor of relatively high losing stakes in first-round games would be a

[25]The same scheme exists for females.

[26]Starting from 2016, no points are assigned for Davis Cup ties.

As a rule, players get paid for their participation. This amount covers traveling costs and other supplementary expenses. However, the payment cannot be counted as a prize because it does not generate incentives to compete.

[27]In this respect the proposed application has a clear advantage over experimental tests one can run in the lab where participants need time to learn rules of the game.

[28]See http://www.espn.com/tennis/aus13/story/_/id/8843972/players-say-prize-money-increase-australian-open-positive-step.

participation story. Nevertheless, this does not seem to be a single driving force in case of highly prestigious contests:

1. Four Grand Slam tournaments do not overlap in time. As a result, they do not need to compete for players.[29] Moreover, the contests have very similar sets of participants and comparable prize schedules. In particular, the allocation of points is identical for all Grand Slams.

2. Top–players who are likely to advance in the contest are always willing to participate. First, winning the Grand Slam gives 2,000 ranking points when other tournaments pay at most 1,000. Second, significant monetary rewards allocated in the finals induce both participation and effort exertion. Since best athletes win with a higher probability, they always prefer entering the competition to abstention.

3. Weaker athletes must compete in qualifying games to get one out of 16 places in the main draw of the Grand Slam. Thus, participation is competitive.

4. The Australian Open compensates traveling and living costs of the participants.[30] Thus, positive monetary prizes for first-round losers cannot be seen as a way to reimburse these expenses.

Although the participation story is potentially important, it cannot completely explain how higher losing prizes affect effort choices.

1.4.1 The Data

I analyze the Australian Open and the French Open in 2009–2015.[31] Only first–round matches for male players are taken into account. There are several reasons to restrict the dataset in such a way. I do not look at later stages of the contests because one can treat losing rewards assigned there as the prize for being advanced. However, I account for these prizes when compute first-round winning benefits (more details will follow). Second, in 2009 the ATP changed the ranking points allocation between stages of the Grand Slams. Thus, years before 2009 are excluded as an attempt to isolate possible structural breaks. Also, a relatively short horizon (2009–2015) allows me to assume stable preferences over time.

[29]In fact, the time interval between the Australian Open and the nearest Grand Slam is about 3 months. This gives players enough time to recover and prepare for the next big competition.

[30]See http://www.espn.com/tennis/aus13/story/_/id/8843972/players-say-prize-money-increase-australian-open-positive-step.

[31]The data are available online at https://matchstat.com/.

As mentioned before, in 2013, the Australian Open increased first-round monetary losing prizes significantly (up to 40% compared to the previous year).[32] The policy might have shifted players' preferences or made them contest-specific. Given that the Australian Open and the French Open have similar sets of participants, I use the latter competition as a control group for the former one. Also adding another comparable tournament makes a set of observed prizes and players' responses richer.

Another argument in favor of choosing the French Open but not another Grand Slam as a control group is the contest's position in the annual tournament schedule. In fact, the French Open (end of May) follows the Australian Open (mid-January). Thus, both competitions take place in the first half of the game season. Given that the ATP World Tour Finals played by top–8 athletes are held in November, contestants' preferences may change when the end of the season is approaching. If one takes the Wimbledon (late June) or the US Open (late August), stronger players willing to qualify as top–8 can shift their preferences in favor of non-monetary prizes. As a result, contestants' behavior might evolve once the game season proceeds. In this respect, the French Open is the best control group to avoid inconsistencies in preference estimates.

I exclude female players from the sample for two reasons:

1. In 2014, the WTA changed the ranking points allocation between stages of the Grand Slams. The policy targeted only females.[33] The event could shift players' preferences, and for now I prefer to isolate this source of variation.

2. In Grand Slam events, females play according to the "best–of–3" scheme, but males follow the "best–of–5" rule.[34] The two settings may shape the athletes' behavior differently. Moreover, with the "best–of–3" scenario, the effect of luck or other random factors on the contest outcome might be stronger. Hence, it is potentially problematic to get consistent estimates of skills and preferences in the econometric model where females and males are analyzed together.

Further, one have to find a good proxy for contestants' effort. Most statistics reported in tennis matches are relative.[35] However, to estimate the model, absolute effort must be approximated. I solve this issue by constructing a measure based on a number of unforced errors:

> "An unforced error is when the player has time to prepare and position himself or herself to get the ball back in play and makes an error. This is a shot that the player would normally get back into play." [36]

[32]See Tables 1.6 and 1.7.

[33]Male and female professional tennis players belong to different associations. These two entities fix the points' allocation over annual tournaments and calculate athletes' rankings independently.

[34]In Grand Slams, females and males play at most 3 and 5 sets per match, respectively.

[35]For example, a number of points won in a particular match depends on the actions of both players.

[36]See http://tt.tennis-warehouse.com/.

By definition, the unforced error is not the outcome of a direct strategic interaction between competitors.[37] Timothy W. Gallwey, a professional tennis coach, says in one of his books, "When I was playing tournament tennis and made an unforced error, I understood that the actual cause of my error was a momentary lapse in focus. I let the error be a wake-up call to come back to the present moment."[38] The claim relates unforced errors to lower effort exertion. One may argue such mistakes are driven by risk-taking or characterize a player-specific game style. These concerns are potentially important, and I will address this point later.

Let u_{ij} be a number of unforced errors player i makes in match j (see Table 1.8 for summary statistics). Since longer games are likely to show more unforced errors, I weight u_{ij} by a number of sets played (ns_j) and define $\tilde{u}_{ij} = \frac{u_{ij}}{ns_j}$. Finally, assume there exists a continuous function h (identical for all contestants) that transforms individual effort (e_{ij}) into $\tilde{u}_{ij}$, $\frac{\partial h(e_{ij})}{e_{ij}} < 0$. In other words, higher e_{ij} translates into less unforced errors per set played. For simplicity, take $h\left(e_{ij}\right) = \hat{u} - \tilde{e}_{ij}$ where $\hat{u} = \max_{i,j} \tilde{u}_{ij}$.[39]

In Appendix B, I show that the proposed measure is a good proxy for effort. Standard contest models with one-dimensional prizes deliver two predictions:

1. Effort must increase in the prize spread.

2. Effort must decrease in contestants' heterogeneity.

Reduced-form non-experimental tests of the contest theory conducted with different effort measures strongly support these hypotheses.[40] The same holds for the proxy based on a number of unforced errors per set. This allows me to conclude that the current measure performs at least as good as the alternatives used in the previous work. Further, I investigate if the proxy includes a risk-taking component. To perform the test, I assume that contestants who choose riskier strategies tend to serve with a higher speed. Then, if the number of unforced errors per set also includes risk-taking, the two variables must be positively correlated. Table 1.17 shows that this hypothesis must be rejected.

Next, I check the properties of the points' empirical distribution (see Figure 1.5.1). Only 1% of players hold more than 10,000 ranking points. At the same time, around 70% have 1,000 points or less. In fact, the distribution is very skewed towards the left. To capture this feature, players

[37]Paserman (2010) also looks at a number of unforced errors to measure the performance of tennis players. He claims this approach makes an improvement over previous empirical studies on tennis that approximated effort as the number of games or sets won.

[38]Gallwey, W.T. (1981). "The Inner Game of Golf." 1st ed. New York: Random House.

[39]In principle, one could come with more complicated specifications of h. However, given unobservability of effort, there would be no way to achieve identification. For this reason I stick to the simplest possible functional form that supports $\frac{\partial h(e_{ijk})}{e_{ijk}} < 0$.

[40]For example, see Maloney and McCormick (2000), Sunde (2003), Sunde (2009), Brown (2011), Berger and Nicken (2014).

are divided into groups based on their ranking points. Namely, I introduce variable q that shows which quantile of the points' empirical distribution a contestant belongs to. This classification is important because various groups may attach different valuations to the points dimension.[41]

Definition 1.3. *Let D be a strictly ordered set of threshold values (d) with at least two elements ($\#D \geq 2$). Player i belongs to group k ($q_i = k$) if and only if*

$$points_i \in [d_{\#D-k},\, d_{\#D-k+1})$$

where $points_i$ denotes a number of ranking points player i holds, $k = \{1, ..., \#D - 1\}$.[42]

Other relevant variables include:[43]

1. Tournament-specific controls: dummies for years 2009–2015, fixed effects of the two contests, reward schedules, and the total prize money.

2. Players' characteristics: age, the body mass index (BMI), home bias, a number of ranking points, and whether a contestant was seeded or not.

3. Match-specific features: approximated heterogeneity between players, a binary variable that indicates if a game took place on Day 1 or Day 2.[44]

Finally, I calculate the following empirical moments and use them to evaluate the predicting power of the structural model:

- Average winning probabilities for two contestant types (i.e. g and n) and seeded players;

- Average winning probabilities for different age and ranking groups.

In the theoretical model, I defined the "greediest-to-win" (g) and not the "greediest-to-win" (n) types based on players' skill-valuation profiles. Obviously, this cannot be directly observed in the data. Nevertheless, I propose two ways to approximate contestants' identities:

[41] One additional point can be of higher importance for players with lower ranks.

[42] In the structural model I use the following specification of D:

$$D = \{0,\, q\,(.15)\,,\, q\,(.4)\,,\, q\,(.6)\,,\, q\,(.75)\,,\, q\,(.85)\,,\, q\,(.92)\,,\, q\,(.97)\,,\, q\,(.99)\,,\, q\,(1)\}$$

where $q\,(p)$ corresponds to a p-quantile of the points' empirical distribution. Coarser partitions worsen the goodness–of-fit. This happens because they lump very heterogeneous players together.

[43] Summary statistics can be found in Table 1.8.

[44] All first-round matches are scheduled for first two days of Grand Slam tournament.

1. Player i is treated as the "greediest-to-win" in match j ($g_j = i$) if he holds more ranking points than the competitor:

$$g_j = i \Leftrightarrow points_{i,\,j} > points_{-i,\,j}$$

The ATP rankings include all tournaments where players performed in last 52 weeks. Thus, the previous success must map into stronger preferences over winning and / or better skills.

2. Player i is treated as the "greediest-to-win" in match j ($g_j = i$) if his winning probability derived from betting odds ($p^B_{i,\,j}$) exceeds the one of the opponent:

$$g_j = i \Leftrightarrow p^B_{i,\,j} > p^B_{-i,\,j}$$

Betting odds can aggregate more information than just ranking points. As a consequence, this approach may identify contestants' types better than the previous one.

The partition based on contestants' age looks as follows:

$$d^{age}_k \in D^{age},\; D^{age} = \{\min_i (age_i)\,,\, 20,\, 25,\, 30,\, \max_i (age_i)\}$$
$$i \in k \Leftrightarrow age_i \in \left[d^{age}_k,\, d^{age}_{k+1}\right)\,,\; k = \{1,\, ...,\, 4\}$$

where k is the group. Variable q defines the partition driven by ranking points.

1.4.2 Structural Estimation

1.4.2.1 The Model Setup

To recover contestants' skill-valuation profiles, I explore the variation in monetary prize schedules across tournaments / years, observed effort choices, and player-specific controls. The proposed theoretical model delivers clear predictions that can be validated empirically. I formulate three results that map underlying skills and preferences into realized effort and match outcomes given the observed variation in a prize scheme.[45] Suppose the winning benefit in dimension A grows at the cost of the losing reward, i.e. the corresponding prize spread ($\triangle_A$) gets larger.

Result 1. For any match j, a skill-valuation profile is such that expected aggregate effort decreases in $\triangle_A$ (given $\triangle_B$) if and only if

1. The expected effort of player g (n) increases (decreases) in $\triangle_A$, and the latter effect dominates:

$$\frac{\partial E_g(\cdot)}{\partial \triangle_A} > 0,\; \frac{\partial E_n(\cdot)}{\partial \triangle_A} < 0\, and\, \left|\frac{\partial E_g(\cdot)}{\partial \triangle_A}\right| < \left|\frac{\partial E_n(\cdot)}{\partial \triangle_A}\right|$$

[45] See Section 1.3 for technical details.

2. The expected probability that player g (n) wins increases (decreases) in $\triangle_A$:

$$\frac{\partial P_g^{win}(\cdot)}{\partial \triangle_A} > 0,\ \frac{\partial P_n^{win}(\cdot)}{\partial \triangle_A} < 0$$

Result 2. For any match j, a skill-valuation profile is such that expected aggregate effort increases in $\triangle_A$ (given $\triangle_B$) if and only if

1. It is either the expected effort of both players increases in $\triangle_A$

$$\frac{\partial E_i(\cdot)}{\partial \triangle_A} > 0,\ i = \{g,\, n\}$$

or the expected effort of player g (n) increases (decreases) in $\triangle_A$, and the former effect dominates:

$$\frac{\partial E_g(\cdot)}{\partial \triangle_A} > 0,\ \frac{\partial E_n(\cdot)}{\partial \triangle_A} < 0\, and \left|\frac{\partial E_g(\cdot)}{\partial \triangle_A}\right| > \left|\frac{\partial E_n(\cdot)}{\partial \triangle_A}\right|$$

2. The expected probability that player g wins either (weakly) increases or decreases in $\triangle_A$:

$$\frac{\partial P_g^{win}(\cdot)}{\partial \triangle_A} \geq 0 \Leftrightarrow \frac{\alpha_g}{\beta_g} \geq \frac{\alpha_n}{\beta_n}\ or\ \frac{\partial P_g^{win}(\cdot)}{\partial \triangle_A} < 0 \Leftrightarrow \frac{\alpha_g}{\beta_g} < \frac{\alpha_n}{\beta_n}$$

Result 3. For any match j, a skill-valuation profile is such that contestants are not sensitive to incentives in dimension B if and only if[46]

1. The expected effort of both players increases in $\triangle_A$:

$$\frac{\partial E_i(\cdot)}{\partial \triangle_A} > 0,\ i = \{g,\, n\}$$

2. The expected probability that player g wins stays constant when $\triangle_A$ grows:

$$\frac{\partial P_g^{win}(\cdot)}{\partial \triangle_A} = 0$$

The proposed identification strategy employs these theoretical results, the variation in prize schedules, approximated contestants' effort / types, and the information about winning players to restore different skill-valuation profiles (other identification assumptions will be discussed later).

Let money and ranking points (career concerns) be labeled as goods A and B, respectively. To bring the theoretical framework to the data, I introduce a setup similar to a random utility

[46]The opposite holds when players do not respond to incentives in dimension A.

model.[47] Suppose contestants get additional individual-specific utility (or disutility) when lose:[48]

$$U_i^L\left(A^L,\, B^L,\, \varepsilon_i\right) = c_i\left(\alpha_i A^L + \beta_i B^L\right) + \varepsilon_i,\, \varepsilon_i \sim F_i(\varepsilon)\,,\, \varepsilon \in [\underline{\varepsilon}_i,\, \bar{\varepsilon}_i]$$

where $F_i(\varepsilon)$ is a truncation of a mean zero normal distribution with a standard deviation σ_ε:[49]

$$f_i(\varepsilon) = \frac{\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\bar{\varepsilon}_i}{\sigma_\varepsilon}\right) - \Phi\left(\frac{\underline{\varepsilon}_i}{\sigma_\varepsilon}\right)},\, \varepsilon \in [\underline{\varepsilon}_i,\, \bar{\varepsilon}_i]$$

I assume the shocks are independently drawn from the underlying player-specific distributions. Suppose both competitors, but not an econometrician, observe ε_i.[50] All Grand Slams match players at random in first-round games. Then, one can treat idiosyncratic ε_i as the utility (or disutility) of losing against a particular opponent.[51] With this approach, contestants' types become random:

$$\tilde{t}_i = c_i\alpha_i\triangle_A + c_i\beta_i\triangle_B - \varepsilon_i = t_i - \varepsilon_i$$

where $\triangle_A$ and $\triangle_B$ specify the difference between winning and losing prizes in corresponding dimensions. In particular, $\tilde{t}_i$ is drawn from a truncated normal distribution (a linear transformation of $F_i(\varepsilon)$) and has the following characteristics: [52]

$$E\left(\tilde{t}_i\right) = t_i - E(\varepsilon_i)\,,\, Var\left(\tilde{t}_i\right) = Var(t_i - \varepsilon_i)$$

Player i is the "greediest-to-win" ($\tilde{t}_i = \tilde{t}_g$) if and only if $\tilde{t}_i > \tilde{t}_{-i}$:

$$\tilde{t}_i > \tilde{t}_{-i} \Leftrightarrow \varepsilon_i - \varepsilon_{-i} < t_i - t_{-i}$$

The proposed theoretical model restricts the support of contestants' strategies (see Section 1.3). Specifically, their effort levels cannot exceed $\min\left\{\tilde{t}_i,\, \tilde{t}_{-i}\right\}$:

$$e_i \le \min\left\{\tilde{t}_i,\, \tilde{t}_{-i}\right\} = \min\left\{t_i - \varepsilon_i,\, t_{-i} - \varepsilon_i\right\} \equiv \tilde{t}_n$$

[47] One can find similar settings in applied studies of auctions or industrial organization. Bajari et al. (2010) provide the profound overview of modern approaches one can use to recover underlying parameters of static and dynamic games with different information structures. Authors indicate that often structural inference involves conditioning on reduced forms. As a result, the estimation becomes equivalent to a single-agent problem. Donald and Paarsch (1996), Bajari and Hortacsu (2003) work with the auction setting and formulate a parametric likelihood estimator based on players' winning probabilities. Degan (2007), Degan and Merlo (2011) provide coherent examples of the likelihood construction in the individual choice setting where player-specific characteristics (age, education etc.) matter.

[48] The additive separable noise must be introduced in an asymmetric way. If it has the same effect on winning and losing utility, ε_i cancels out when one defines contestants' types (t_i). Since winning probabilities used to construct the likelihood function depend on t_i, the symmetric noise setting becomes deterministic.

[49] See Appendix C for the discussion of the player-specific noise distribution.

[50] I justify non-observability of ε_i for the econometrician as follows. Contestants have much more information about idiosyncrasies affecting their opponents (training regime, health- and career-related concerns, other specific circumstances etc.). Moreover, they can see each other's play in different competitions and draw conclusions about the game style and tactical steps a particular individual uses. All these things may shape contestants' perception of losing. However, they are hard to observe and aggregate for the econometrician.

[51] For example, take the contestant ranked 100 who can lose against either one of top–5 athletes or a player with similar characteristics. In the former case, he may feel less discouraged and even perceive this event as a valuable experience.

[52] As Appendix C shows, $E\left(\tilde{t}_i\right)$ and $Var\left(\tilde{t}_i\right)$ depend on player- and competitor-specific controls.

The latter inequality defines a player-specific upper bound on ε_i ($\bar{\varepsilon}_i$) that depends on the parameters of interest (namely, contestants' skills and preferences). Appendix C specifies other conditions the distribution of ε_i must satisfy to match the theoretical framework. Thus, the econometric model is characterized by parameter-dependent support. This feature has important implications for the estimation procedure, and I discuss them later.[53]

Let π and x denote the sets of parameters and controls, respectively. Given that players' skills (c) interact with their preferences (α and β) in a multiplicative way, one cannot identify these values separately without imposing additional restrictions. Define a joint effect of c and α (β) as $\tilde{\alpha}$ ($\tilde{\beta}$) and express it as a function of π and x:

$$\tilde{\alpha}\left(\pi_{\tilde{\alpha}},\, x_{\tilde{\alpha}}\right) \equiv c\left(\pi_c,\, x_c\right)\alpha\left(\pi_\alpha,\, x_\beta\right)$$

$$\tilde{\beta}\left(\pi_{\tilde{\beta}},\, x_{\tilde{\beta}}\right) \equiv c\left(\pi_c,\, x_c\right)\beta\left(\pi_\beta,\, x_\beta\right)$$

where π_l and x_l reflect subsets of π and x, respectively, that shape a value of l, $l = \left\{c,\, \alpha,\, \beta,\, \tilde{\alpha},\, \tilde{\beta}\right\}$. Estimating $\tilde{\alpha}(\cdot)$ and $\tilde{\beta}(\cdot)$ is sufficient to capture the relative importance of different prize dimensions for player i (i.e. $\frac{\tilde{\alpha}\left(\pi_{\tilde{\alpha}},\, x^i_{\tilde{\alpha}}\right)}{\tilde{\beta}\left(\pi_{\tilde{\beta}},\, x^i_{\tilde{\beta}}\right)}$) and measure the degree of heterogeneity between the competitors.

To identify $\tilde{\alpha}$ and $\tilde{\beta}$, I exploit the variation in player- and tournament-specific characteristics. The main challenge is to find the instruments to isolate the effects of $\tilde{\alpha}$ and $\tilde{\beta}$. I impose the following functional assumptions: [54]

$$c\left(\pi_c,\, x_c^{ijt}\right) = exp\left(\pi_1^c points_i + \pi_2^c age_i + \pi_3^c age_i^2 + \pi_4^c BMI_i + \pi_5^c hbias_i + \pi_6^c Tour + \pi_Z^c Z_{jt}\right)$$
$$\tilde{\alpha}\left(\pi_{\tilde{\alpha}},\, x_{\tilde{\alpha}}^{ijt}\right) = c\left(\pi_c,\, x_c^{ijt}\right) exp\left(\pi_1^{\tilde{\alpha}} + \pi_2^{\tilde{\alpha}} seed_i\right)$$
$$\tilde{\beta}\left(\pi_{\tilde{\beta}},\, x_{\tilde{\beta}}^{ijt}\right) = c\left(\pi_c,\, x_c^{ijt}\right) exp\left(\pi_1^{\tilde{\beta}} + \pi_2^{\tilde{\beta}} q_i\right)$$

where:

- $points_i$ is a number of ranking points contestant i holds;
- BMI_i reflects the body mass index;
- $hbias_i = 1$ if i plays in his home country and $hbias_i = 0$, otherwise;
- $Tour = 1$ for the Australian Open and $Tour = 0$ for the French Open;
- Z_{jt} represents a set of dummy variables that includes:

[53] For example, see Hirano and Porter (2003).
[54] The exponents guarantees non-negativeness of skills and preference parameters.

  - A day when match j took place ($d_j = 1$ if match j was schedules for Day 1; otherwise, $d_j = 0$);[55]
  - A year-specific effect (t);
  - Interaction terms $Tour \times d_j$ and $d_j \times t$.[56]

- $seed_i = 1$ if i was seeded in the tournament draw and $seed_i = 0$, otherwise;[57]
- q_i reflects to which quantile of the points' empirical distribution player i belongs to (see Subsection 1.4.1).

Obviously, a number of ranking points ($points_i$) conveys the information about players' skills. Intuitively, top-athletes have more experience and display better game statistics than their lower-ranked peers. Generally, players perform worse in the beginning and closer to the end of their career. To capture this U-shaped pattern, I include age_i and age_i^2 into the $c(\pi_c, x_c^i)$ specification. Since tennis is a physically intensive game, athletes with bigger BMI_i values may have an advantage in serve speed and win more often. On the other hand, higher BMI_i can result in slower running. Thus, the effect of BMI_i on athletes' performance is not ex-ante clear. Contestants playing in their home countries may know the courts better, feel more support, avoid acclimatization. These factors will work in their favor and improve the skills. Variables $Tour$ and Z_{jt} are included to account for contest- and time-specific shocks all players face (a type of the surface, geographical location, weather conditions etc.).

It is assumed that individual preferences over prize items (namely, $\tilde{\alpha}(\cdot)$ and $\tilde{\beta}(\cdot)$) are imperfectly correlated with contestants' rankings. Top–32 seeded players ($seed_i = 1$) have already advanced in their career and earned a significant amount of prize money. Consequently, their preferences over this good can differ from those lower-ranked competitors display.[58] Hence, $seed_i$ must affect $\tilde{\alpha}(\cdot)$. Next, $\tilde{\beta}(\cdot)$ (a valuation attached to a non-monetary dimension) has to depend on contestant i's position in the points' empirical distribution captured by the q_i variable. For example, one additional point can be more important for players with lower ranks because it grants them access to better contests and improves career prospectives.[59] The observed variation in monetary prize schedules across tournaments and years, coupled with fixed rewards in the non-monetary dimension, also helps in isolating the effect of $\tilde{\alpha}(\cdot)$ from $\tilde{\beta}(\cdot)$.

[55] All first-round matches take place on first two days of the tournaments.

[56] I do not include $Tour \times t$ into Z_{jt} because it perfectly correlates with contest-specific monetary prize schedules.

[57] Every Grand Slam has 32 seeded contestants (25% of all players) who never meet each other in the first round. These are top-ranked athletes with enough experience and a high probability of being advanced in the tournament.

[58] For example, a marginal value of money can be higher for non-seeded players who, in case of winning the prize, get a chance to invest in their training and perform better in the future.

[59] On the contrary, top-players may be more career-concerned, which corresponds to higher values of β's.

The way Grand Slams match players and the modeling assumptions imposed on $\tilde{\alpha}(\cdot)$ and $\tilde{\beta}(\cdot)$ make it possible to identify the σ_ε parameter (the variance of the underlying noise distribution). First, the fact that players are randomly paired excludes any kind of selection bias driven by the strategic choice of an opponent. Second, adding the $Tour$ variable and a set of time- and tournament-specific dummies, together with their interaction terms (Z_{jt}), allows me to isolate common shocks contestants face. Hence, all the variance left can be attributed to a random noise term.

I characterize first-round winning prizes as a continuation value of being advanced in the contest. Suppose success and failure at later stages of the tournament are equally probable. Also, assume that identities of potential future opponents do not matter.[60] With this approach, prize spreads are equal for all players. As a result, heterogeneity in first-round games stems only from different skills and asymmetric preferences.

To estimate the model, I treat matches as independent and formulate the likelihood in terms of players' winning and losing probabilities. Let P_{gj} (P_{nj}) denote a probability that type g (n) wins match j. Since player n stays inactive with probability $p_n^0 \in (0,\, 1)$, P_{ij} must include two components – P_{ij}^1 and P_{ij}^2 (see Proposition 1.1):

1. With probability p_n^0, type g certainly wins:
$$P_{gj}^{*}\left(\pi,\, \varepsilon_{gj},\, \varepsilon_{nj}\right) = p_n^0,\; P_{nj}^{*}\left(\pi,\, \varepsilon_{gj},\, \varepsilon_{nj}\right) = 0$$

2. Otherwise, type g (n) wins with probability $\left\{\frac{e_{gj}}{\tilde{t}_{nj}}\right\}$ ($\left\{\frac{e_{nj}}{\tilde{t}_{nj}}\right\}$):
$$P_{gj}^{**}\left(\pi,\, \varepsilon_{gj},\, \varepsilon_{nj},\, e_{gj}\right) = \left(1-p_n^0\right)\frac{e_{gj}}{\tilde{t}_{nj}},\; P_{nj}^{**}\left(\pi,\, \varepsilon_{gj},\, \varepsilon_{nj},\, e_{nj}\right) = \left(1-p_n^0\right)\frac{e_{nj}}{\tilde{t}_{nj}}$$

   where e_j^g (e_j^n) corresponds to observed effort exerted by type g (n) in match j.

Then, contestant i's winning probability can be calculated as a sum of the two components, $P_{ij}^{*}(\cdot)$ and $P_{ij}^{**}(\cdot)$:
$$P_{ij}\left(\pi,\, \varepsilon_{gj},\, \varepsilon_{nj},\, e_j^i\right) = P_{ij}^{*}\left(\pi,\, \varepsilon_{gj},\, \varepsilon_{nj}\right) + P_{ij}^{**}\left(\pi,\, \varepsilon_{gj},\, \varepsilon_{nj},\, e_j^i\right)$$

and a complete data likelihood for match j looks as follows:[61]
$$\begin{aligned} L_j^c\left(\pi \,|\, W_{1j},\, e_{1j},\, e_{2j},\, x_1,\, x_2,\, \varepsilon_{1j},\, \varepsilon_{2j}\right) = &\\ I\{1=g\}\left[\left[P_{gj}^1 \cdot \left(1-P_{nj}^2\right)\right]^{I\{W_{1j}=1\}}\left[\left(1-P_{gj}^1\right)\cdot P_{nj}^2\right]^{1-I\{W_{1j}=1\}}\right] + &\\ \left(1-I\{1=g\}\right)\left[\left[P_{nj}^1 \cdot \left(1-P_{gj}^2\right)\right]^{I\{W_{1j}=1\}}\left[\left(1-P_{nj}^1\right)\cdot P_{gj}^2\right]^{1-I\{W_{1j}=1\}}\right] & \end{aligned}$$

---

[60]This strategy was used in several empirical papers (for instance, see Silverman and Seidel (2011), Ivankovic (2007)) with the following argument: winning and losing probabilities of heterogeneous players average to "1/2–1/2".

[61]This formulation is similar to a standard probit / logit model.

where

- $P^i_{gj} \equiv P_{gj}\left(\pi,\, \varepsilon_{1j},\, \varepsilon_{2j},\, e^i_j\right)$ and $P^i_{nj} \equiv P_{nj}\left(\pi,\, \varepsilon_{1j},\, \varepsilon_{2j},\, e^i_j\right)$ for $l = \{1,\, 2\}$;
- $W_{1j} = 1$ if player 1 wins match j;
- $I\{1 = g\} = 1$ if player 1 is of type g;
- x_1 and (x_2) reflect individual characteristics of players 1 and 2, respectively.

With this likelihood specification, I use the information on both realized effort levels, e_{1j} and e_{2j}, that are generally not equal.[62] Exploiting the variation in e_{1j} and e_{2j} helps me to recover supports of the contestants' strategies, which depend on $\tilde{t}_{nj}$, and identify the parameters shaping $\tilde{\alpha}(x)$ and $\tilde{\beta}(x)$.

Since ε_{1j} and ε_{2j} are unobservable, one must calculate a probability that contestant 1 is the "greediest-to-win" in match j:

$$P^g_{1j}(\pi,\, x_1,\, x_2) \equiv P(1 = g) = \int_{\underline{u}_j}^{t_1 - t_2} f_j(u)\, du,\ \underline{u}_j = \underline{\varepsilon}_{1j} - \bar{\varepsilon}_{2j}$$

where $u_j = \varepsilon_{1j} - \varepsilon_{2j}$ and $t_i = \tilde{\alpha}(x_i)\triangle_A + \tilde{\beta}(x_i)\triangle_B,\ i = \{1,\, 2\}$. Replacing $I\{1 = g\}$ with $P^g_{1j}(\pi,\, x_1,\, x_2)$ and taking conditional expectations over ε_{1j} and ε_{2j}, I formulate an expected likelihood:

$$L^e_j(\pi \mid W_{1j},\, e_{1j},\, e_{2j},\, x_1,\, x_2) =$$
$$P^g_{1j}(\pi,\, x_1,\, x_2)\, E_{\varepsilon_{nj},\varepsilon_{gj}}\left[\left[P^1_{gj}\cdot\left(1 - P^2_{nj}\right)\right]^{\{W_{1j}=1\}}\left[\left(1 - P^1_{gj}\right)\cdot P^2_{nj}\right]^{1 - I\{W_{1j}=1\}} \mid 1 = g\right] +$$
$$\left[1 - P^g_{1j}(\pi,\, x_1,\, x_2)\right] E_{\varepsilon_{nj},\varepsilon_{gj}}\left[\left[P^1_{nj}\cdot\left(1 - P^2_{gj}\right)\right]^{I\{W_{1j}=1\}}\left[\left(1 - P^1_{nj}\right)\cdot P^2_{gj}\right]^{1 - I\{W_{1j}=1\}} \mid 1 = n\right]$$

When one takes a logarithm of $L^e_j(\pi \mid W_{1j},\, e_{1j},\, e_{2j},\, x_1,\, x_2)$ and aggregates over K matches, the objective function to maximize becomes:

$$logL^e(\pi \mid W,\, e,\, x) = \sum_{j=1}^{K} log\left[L^e_j(\pi \mid W_{1j},\, e_{1j},\, e_{2j},\, x_1,\, x_2)\right]$$

Finally, I can write down a complete optimization program with a single constraint:

$$logL^e(\pi \mid W,\, e,\, x) \to \max_\pi$$
$$s.t.\ \max\{e_{1j},\, e_{2j}\} \le \min\left\{U^W_{gj},\, U^W_{nj}\right\}\ \forall\, j = \{1...K\}$$

where the inequality guarantees well-defined supports for player-specific noise distributions (see Appendix C). Since the constructed econometric model features parameter-dependent support, it

[62]If $e^i_j = e^{-i}_j$, it must be $P_{gj}(\cdot) = (1 - P_{nj}(\cdot))$. The same holds if one focuses on expected winning probabilities and does not use the information on realized effort for both players.

violates a standard regularity condition of the maximum likelihood estimation. Specifically, the solution of the optimization program does not need to be interior. On top of this, the estimator is not necessarily asymptotically normal. To address these issues, I develop the following approach. First, a derivative-free numerical optimization routine that allows for corner solutions is employed to solve the program. Second, to avoid any results that rely on the assumption of asymptotic normality, I compute bootstrap standard errors. To study the properties of the estimator in more detail, I run simulations.[63] Despite the indicated non-regularity, the estimator is consistent. However, as Hirano and Porter (2003) point out, it can be asymptotically inefficient and must be improved later on.[64]

1.4.2.2 Results and the Goodness–of–Fit

In the beginning, I follow the baseline and estimate the distribution of contestants' types for the two-dimensional model (see Table 1.1, column 1). Most of the variables are statistically significant. As expected, more ranking points map into better skills. Importantly, non-monetary incentives matter for effort provision: the respective regression coefficients, $\pi_1^{\tilde{\beta}}$ and $\pi_2^{\tilde{\beta}}$, are statistically significant (see Table 1.1, column 1). Higher ranks affect contestants' preferences over two prize dimensions in the following way. Being seeded ($seed_i = 1$) decreases the marginal utility of money. At the same time, top-players (those with lower values of q_i) tend to value the non-monetary dimension more and display stronger career concerns.

The presence of home bias has a positive effect on athletes' performance. This finding is in line with the previous empirical tests of the contest theory. On average, individuals play better when the body mass index declines. Skills indeed turn to be non-linear in contestants' age: both age_i and age_i^2 are statistically significant (see Table 1.1, column 1). Given that the coefficient in front of age_i^2 is negative, one can approximate when the skills achieve their maximum over age (keeping other player-specific characteristics constant):

$$age^* = argmax_{age_i}\left[ln\left\{c\left(\pi_c,\, x_c^{ijt}\right)\right\}\right] \approx 27.5$$

Finally, contestants tend to perform better in the Australian Open. This effect can be explained as follows. The French open is played on clay courts, while the Australian Open uses hard ones. On average, the athletes find it easier to compete on the latter surface. As a result, one can expect stronger skills in the Australian Open.

Estimated heterogeneity is summarized in Table 1.9. On average, the Australian Open features more uneven matches than the French Open. However, this pattern can be driven by the aforementioned contest-specific effect. For example, take a match from the French Open with

[63] Simulation results are available by request

[64] The lack of asymptotic efficiency is the least problematic property that generally results in bigger standard errors.

heterogeneity H^{FO}. If the same couple plays in the Australian Open, H^{FO} must be multiplied by $e^{\pi_6^c} \approx 1.08$, and $H^{AO} > H^{FO}$ (see Table 1.1 for the estimates). Thus, *ceteris paribus*, heterogeneity is stronger in the latter competition.

Table 1.1: Structural Estimation: Results

Parameter	Both Goods	Only A
$\pi_1^{\tilde{\alpha}}$	–25.709***	.662***
	(3.939)	(.157)
$seed_i$	–18.899***	–.483***
	(2.311)	(.158)
$\pi_1^{\tilde{\beta}}$	.784***	–
	(.152)	
q_i	–2.895***	–
	(.079)	
$points_i$	1.994**	6.988***
	(.906)	(1.528)
age_i	4.473***	–3.345***
	(.219)	(.136)
age_i^2	–4.502***	2.007***
	(.851)	(.328)
BMI_i	–.389**	–.837***
	(.155)	(.132)
$hbias_i$	.112**	.126
	(.045)	(.203)
$Tour$	.074	.087
	(.051)	(.194)
σ_ε	213.078***	195.143***
	(3.632)	(6.498)
Time- and Tournament-Specific Controls	Yes	
$logL$	–969.27	–1343.54
LR–test	$LR = 748.6 > \chi^2(2)$	
$K = 821$		

Note: individual effort is measured as $\tilde{e}_{ij} = \hat{u} - \tilde{u}_{ij}$ where $\tilde{u}_{ij}$ is a number of unforced errors per set played, $\hat{u} = \max_{i,j}, \tilde{u}_{ij}$. Variables $points_i$, BMI_i, age_i are rescaled:

$$\tilde{w}_i = \frac{w_i - \min(w)}{\max(w) - \min(w)}$$

where w_i ($\tilde{w}_i$) denotes the original (rescaled) value of the variable for player i, w is a vector of w_i. Time- and tournament-specific controls correspond to Z_{jt} specified in Subsection 1.4.2.1. K denotes a number of matches. Bootstrap standard errors are reported in brackets. Values with *, **, and *** correspond to 90%, 95%, and 99% significance.

As an attempt to isolate this channel, I re-calculate heterogeneity assuming no contest-specific effect (see Table 1.10). In this case, matches become more uneven in the French Open. Additionally, a range of estimated heterogeneity in this competition is bigger than in the Australian Open (the pattern also holds with the contest-specific effect). One could explain these observations with matching policies used in the two tournaments. Ideally, players must be paired randomly. However, this does not seem to be the case in some Grand Slams. For instance, in the US Open, top-players systematically get weaker opponents than predicted by random matching.[65] As a result, average heterogeneity may increase. If the French Open (but not the Australian Open) adopts similar practices, this might explain the pattern discovered in the data.

Next, the model is simulated 1'000 times. To evaluate the goodness-of-fit, I calculate moments specified in Subsection 1.4.1 and match them against empirical counterparts. In the beginning, consider simulated and actual winning probabilities for two contestant types (namely, g and n) and seeded players (see Table 1.2). Overall, the model replicates type-specific winning probabilities well. It is successful in explaining the difference in contestants' behavior and captures the gap between P_g^W and P_n^W. Also, the model accounts for the fact that seeded players win more often than non-seeded g-types ($P^W > P_g^W$).

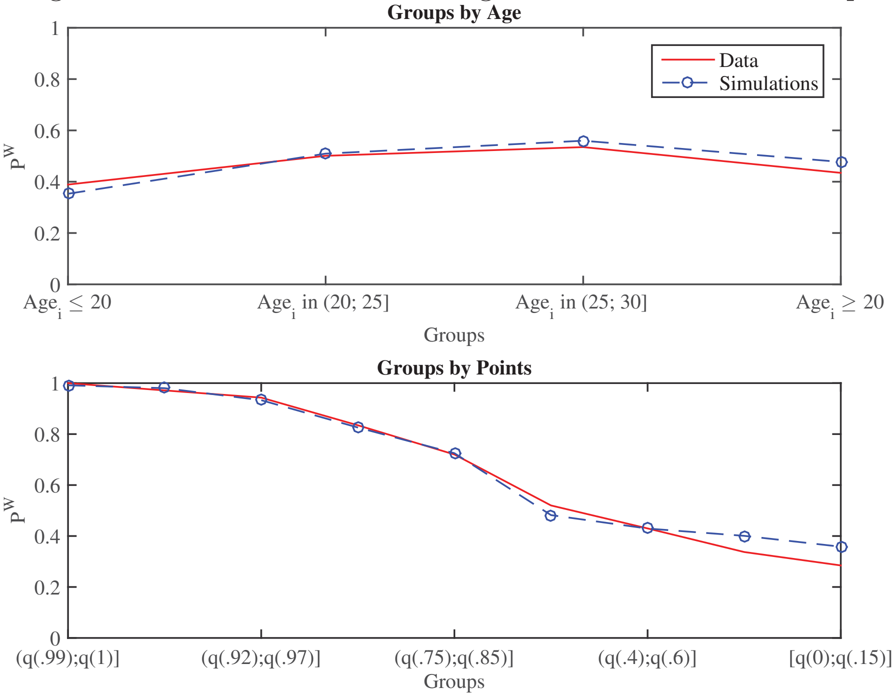
Table 1.2: Simulated vs. Actual Type-Specific Winning Probabilities: Two-Dimensional Model

	The Model	The Data	
		Points–Based Classification	Bets–Based Classification
$\left(P_g^W, P_n^W\right)$	(.721, .279)	(.726, .276)	(.763, .244)
	Seed–Based Classification		
P^W	.824	.844	

Further, I implement the division based on contestants' age and ranking points (see Subsection 1.4.1). Figure 1.4.1 plots simulated and actual winning probabilities for all relevant groups. The age-driven classification manages to match empirical patterns: the two curves are very close to each other. The model predicts the highest winning probability for contestants between 25 and 30. This pattern is in line with the data. The model also replicates an empirical curve under the points-based classification. Actual and predicted winning probabilities are very close in upper quantiles of the points' distribution. The model shows a slight tendency towards overestimating (underestimating) winning probabilities for lower-ranked (middle-ranked) contestants. However, these patterns offset each other.

[65] For example, see http://www.espn.com/espn/otl/story/_/id/6850893/espn-analysis-finds-top-seeds-tennis-us-open-had-easier-draw-statistically-likely.

Figure 1.4.1: Goodness–of–Fit: Age- and Points-Based Groups



In Appendix D, I also check if the structural setup replicates stylized facts observed in players' effort. As before, contestants are divided into age- and points-based groups. In addition, I split the matches by the degree of heterogeneity between competitors. Overall, the model is successful in reproducing shapes of the respective empirical curves (see Figure 1.5.8 and Appendix D for the discussion).

To evaluate the consequences of neglecting one good, I also estimate a specification with monetary prizes only, as always done in the empirical contest literature (see Table 1.1, column 2). The two-dimensional model shows the highest log-likelihood. On top of this, the LR-test rejects a null hypothesis that the baseline specification is equivalent to the single-item alternative.[66]

Table 1.3 shows how well the one-dimensional model replicates type-specific winning probabilities. Overall, it fits the empirical difference in P_g^W and P_n^W much worse than the baseline with two prizes (see Table 1.2). In particular, the model underestimates (overestimates) a probability that type g (n) wins. Also, the one-dimensional setting cannot replicate empirical moments for finer partitions (see Figure 1.5.2). It predicts systematically higher winning probabilities for all age groups and does not distinguish between different elements of the points-based classification.

[66]The testing statistic looks as follows:

$$LR = 2\left(logL^{complete} - logL^{restricted}\right)$$

where $logL^{complete}$ $(logL^{restricted})$ is taken from the two-dimensional model (single-item specification). Given that both restricted models have two parameters less, the statistic must be distributed as $\chi^2(2)$.

Table 1.3: Simulated vs. Actual Type-Specific Winning Probabilities: Only Monetary Prizes

	The Model	The Data	
		Points–Based Classification	Bets–Based Classification
$\left(P_g^W, P_n^W\right)$	(.608, .392)	(.726, .276)	(.763, .244)
	Seed–Based Classification		
P^W	.644	.844	

To understand what drives these effects, I compare the average estimated value of money ($\tilde{\alpha}$) and the corresponding degree of heterogeneity ($H_{\tilde{\alpha}}$) for one- and two-dimensional models (see Table 1.4). First, the single-item setting underestimates heterogeneity in the monetary dimension. This fact explains why the setup generates type-specific winning probabilities that are closer to "1/2–1/2" than observed. Second, it predicts higher valuations attached to money and, consequently, overestimates the incentive effect of this prize. Thus, if one reward component is disregarded, the estimates become biased, and this may lead to incorrect policy recommendations.

Overall, the structural analysis shows that contestants, indeed, respond to multi-dimensional incentives. This finding reveals the directions to improve the existing reduced-form tests of the contest theory that always deal with single-item prizes.

Table 1.4: One vs. Two Dimensions: Estimated Heterogeneity

Two-Dimensional Model	One-Dimensional Model
Mean predicted valuations in dimension A	
.12	1.84
Ratio of mean predicted $H_{\tilde{\alpha}}$ (rescaled): $\frac{H^2_{\tilde{\alpha}}}{H^1_{\tilde{\alpha}}}$	
20.7	
Ratio of predicted $Var\left(H_{\tilde{\alpha}}\right)$ (rescaled): $\frac{Var\left(H^2_{\tilde{\alpha}}\right)}{Var\left(H^1_{\tilde{\alpha}}\right)}$	
6.51	

Note: $H^1_{\tilde{\alpha},j} = \left|\tilde{\alpha}^1\left(x_{ij}\right) - \tilde{\alpha}^1\left(x_{-ij}\right)\right|$ ($H^2_{\tilde{\alpha},j} = \left|\tilde{\alpha}^2\left(x_{ij}\right) - \tilde{\alpha}^2\left(x_{-ij}\right)\right|$) is heterogeneity between contestants in match j in the single-item (two-dimensional) model; to calculate comparable means and standard deviations for the two series, $H_{\tilde{\alpha}}$ is rescaled as follows:

$$h_{\tilde{\alpha},j} = \frac{H_{\tilde{\alpha},j} - \min\left(H_{\tilde{\alpha}}\right)}{\max\left(H_{\tilde{\alpha}}\right) - \min\left(H_{\tilde{\alpha}}\right)}$$

where $H_{\tilde{\alpha}}$ is a vector of $H_{\tilde{\alpha},j}$.

1.4.2.3 Policy Experiments

In this section, I ask if there exist alternative prize schemes that induce more aggregate effort for estimated skill-valuation profiles and the given budget of the contest organizer. In principle, one can abstract from the tennis-specific setting and think about the job promotion context. As it was discussed earlier, ranking points map into career concerns. Thus, the proposed policy experiments have a sufficient degree of generality.

As before, I assume that the designer seeks to maximize expected aggregate effort.[67] In every match j, he can allocate endowments of money and ranking points ($\hat{A}$ and $\hat{B}$, respectively) between winning and losing bundles. Since winners obtain a continuation value of being advanced in the contest, I assume that losing prizes (A^L and B^L) increase (decrease) at cost (benefit) of the final trophy. Then, one can restore the "Winner–Takes–All" schedule as follows:

$$\hat{A} \equiv A_0^W = A_a^W + p_{final} A_a^L$$
$$\hat{B} \equiv B_0^W = B_a^W + p_{final} B_a^L$$

where:

- $\left(A_a^W,\, B_a^W\right)$ and $\left(A_a^L,\, B_a^L\right)$ correspond to actual winning and losing prizes;
- p_{final} is a probability to get to the final.[68]

To run the experiments, I construct grids $\bar{A}^L$ and $\bar{B}^L$ including actual losing prizes:

$$\bar{A}^L = \left[0,\, \bar{u} A_a^L\right],\, \bar{B}^L = \left[0,\, \bar{u} B_a^L\right],\, \bar{u} > 1$$

For every match j and prize scheme k, I simulate the model 1,000 times, calculate expected effort and take the median. At the contest level, a mean over all games is taken.

First, assume **prizes in dimension B (career concerns) are fixed**, i.e. the designer can shape only monetary incentives. Moreover, suppose the **reward schedules are not match–specific**. This setting mirrors the tennis example. As there exists a fixed tournament effect, I analyze the Australian Open and the French Open separately. Importantly, contest organizers never leave good B (ranking points) out because this reduces aggregate effort significantly (see Table 1.11). For this reason, I analyze only two-dimensional prize structures. Figures 1.5.3 and 1.5.4 display how expected effort changes when monetary losing prizes (A^L) grow in both contests. On average, the Australian Open would gain from zero losing benefits in the first round. This effect is driven by responses of relatively strong players whose expected effort decreases in A^L. However, weaker competitors behave differently. Their expected effort

[67] In principle, the designer can have more objectives, especially in the dynamic contest setting. For instance, he may want relatively strong players to advance in the competition with a higher probability.

[68] $p_{final} = \frac{1}{2^6}$ under non-specific prize spreads (the "1/2–1/2" assumption).

increases up to 2.9% (2.7%) with respect to zero (actual) losing prizes when A^L grows (see Figure 1.5.3).

The French Open shows the opposite pattern. On average, the contest would benefit from higher losing prizes in first-round matches. Reallocating 5% of final winning benefits in favor of first-round losing rewards could improve mean expected aggregate effort up to .41% with respect to the actual schedule. Not surprisingly, the effect is relatively small when one must take the average over heterogeneous matches. Nevertheless, the pattern is stable and driven by responses of disadvantaged (type n) contestants (see Figure 1.5.4). In particular, these players improve their effort by more than 3.7% (3.2%) when A^L grows relative to zero (actual) losing benefits. The difference between the Australian Open and the French Open can be explained by more heterogeneous matches in the latter contest (see Subsection 1.4.2.2 for the discussion).

Next, I investigate how players' responses to higher losing prizes depend on the degree of heterogeneity in a particular game. All matches are divided into groups based on the absolute difference in contestants' ranking points (H_j^p):[69]

$$H_j^p = |points_{ij} - points_{-ij}|$$
$$d_k^{het,\,p} \in D^{het,\,p},\ D^{het,\,p} = \left\{0,\ 200,\ 500,\ 1000,\ 1600,\ 3000,\ \max_j\left(H_j^p\right)\right\}$$
$$j \in k \Leftrightarrow H_j^p \in \left[d_k^{het,\,p},\ d_{k+1}^{het,\,p}\right),\ k = \left\{1,\ ...,\ \#D^{het,\,p} - 1\right\}$$

where k is the group, and $\#D^{het,\,p}$ denotes cardinality of $D^{het,\,p}$. Figures 1.5.5 and 1.5.6 depict match-specific responses to growing monetary losing prizes for the two contests. The Australian Open and the French Open show very similar patterns summarized in Table 1.5. Not surprisingly, stronger players (g types) always decrease expected effort when monetary losing prizes grow. On the contrary, the response of their opponents depends on the degree of heterogeneity. In relatively even matches ($H_j^p < 500$), higher losing benefits reduce the expected effort of disadvantaged players. If heterogeneity is strong ($H_j \geq 500$), the pattern changes. Increasing losing prizes improves expected effort of weaker contestants by more than 10% compared to the case of $A^L = 0$ and actual schedules (see Figure 1.5.6). This growth compensates losses caused by negative responses of advantaged players. In particular, reducing final winning prizes by 5% for the benefit of first-round losing rewards increases expected aggregate effort up to 3% in relatively heterogeneous matches (see $H_j^p \in [1000, 3,000)$ in Figures 1.5.5 and 1.5.6).

All observed patterns are in line with the theoretical predictions. Under relatively weak heterogeneity, dominant players do not have a big advantage over their opponents. As a result, the designer faces no need to redistribute a relative power between competitors. Moreover,

[69] I use ranking points but not estimated skill–valuation profiles to approximate heterogeneity because the former is perfectly observable. Also this measure is easier to understand and interpret for contest managers in the real-life setting.

Table 1.5: The Effect of Higher Monetary Losing Prizes and Contestants' Heterogeneity

	$H_j^p < 500$ (44% of matches)	$H_j^p \geq 500$ (56% of matches)
Effect of higher A^L on mean exp. agg. effort	"–"	"+"

winning benefits of both contestant types decline more or less symmetrically when higher losing prizes are introduced. In this case, the positive equilibrium effect will never compensate losses caused by the prize stake reduction. Thus, the "Winner–Takes–All" allocation induces the highest aggregate effort when players have similar skills and preferences.

If one contestant has a very strong advantage in skills or at least in one prize dimension, the opponent feels discouraged. Then, the "Winner–Takes–All" schedule does not induce enough aggregate effort. However, the designer handles this heterogeneity issue by introducing positive losing prizes. The policy makes players' types more balanced (ideally, $t_n = t_g$), and the contest becomes homogeneous. Now, the competitor who used to be disadvantaged has higher chances to win and increases his effort. In case of pronounced heterogeneity, this positive equilibrium effect exceeds losses associated with lower winning benefits. Eventually, the total effort grows.

Next, suppose **the designer can change prizes in both dimensions.** Although this is not the case in the tennis application, the experiment becomes important in a more general job promotion setting. As before, the designer is constrained by fixed endowments of goods ($\hat{A}$ and $\hat{B}$) and uniform reward schedules. Figure 1.5.7 shows how mean expected aggregate effort changes with losing prizes in both contests. Overall, the Australian Open (the French Open) can improve up to .25% (.24%). Key patterns in the monetary dimension are identical to those found in the previous case. On average, the Australian Open benefits from the "Winner–Takes–All" schedule. Alternatively, higher monetary losing prizes ($A^L > A^L_a$) improve mean expected aggregate effort in the French Open. However, both contests would prefer to allocate ranking points only to first-round winners ($B^L = 0$). As a result, the aforementioned difference between the Australian Open and the French Open affects the designers' choices only in the monetary dimension.[70]

Further, I use the partition introduced above and trace group-specific responses to higher losing rewards. As expected, in relatively homogeneous matches ($H_j^p < 500$) the "Winner–Takes–All" schedule provides the strongest incentives to compete. When heterogeneity increases ($H_j^p \geq 500$), positive losing prizes in both dimensions ($A^L > 0$ and $B^L > 0$) improve expected aggregate effort. In this case, the contest becomes more homogeneous, and weaker players get stimuli to compete

[70] In Subsection 1.4.2.2, I explained more heterogeneity in the French Open with the contest-specific seeding policy. Recall that variable $seed_i$ shapes only players' valuations in the monetary dimension (see Subsection 1.4.2.1 for identification restrictions). Thus, experimental findings provide supportive evidence in favor of the matching-based explanation of excessive heterogeneity in the French open.

more aggressively. Tables 1.12 and 1.13 characterize group-specific reward schedules supporting the highest aggregate effort in the Australian Open and the French Open. Surprisingly, actual prize schemes ($W_a = \left(A_a^W,\, B_a^W\right)$ and $L_a = \left(A_a^L,\, B_a^L\right)$) never induce the strongest competition. As before, relatively even pairs ($H_j^p < 500$) perform the best when the "Winner–Takes–All" allocation is implemented. Matches that feature stronger heterogeneity ($H_j^p \geq 500$) require more losing benefits in both dimensions to support the highest expected aggregate effort. In this case, reallocating 5% of money and 2% of ranking points from final winning prizes to first-round losing rewards could improve expected aggregate effort in the French Open (the Australian Open) by more than 4.9% (3.4%). The gain would be 5.6% (3.7%) for the French Open (the Australian Open) compared to the "Winner–Takes–All" schedule. Thus, higher losing rewards improve the contestants' performance in heterogeneous matches.

All experiments conducted in this section can be applied to the job promotion setting directly. When managers decide how to allocate monetary and non-monetary rewards (status, other career-related stimuli etc.) between heterogeneous participants, they may want to assign positive losing prizes. However, if competing workers have similar skills and preferences, the "Winner–Takes–All" scheme induces the highest aggregate effort. In case of multiple bilateral contests with uniform prizes (for example, professional sports), the optimal allocation depends on the pool of potential competitors and the matching policy.

1.5 Conclusion

Job promotion, professional sports and other similar interactions can be seen as bilateral contests with heterogeneous players and rewards including multiple goods. I propose a theoretical framework to model this setting and characterize the optimal prize allocation that maximizes players' aggregate effort. When heterogeneity in preferences and / or skills is strong, the designer must leave a positive prize for a loser. On the one hand, this schedule reduces winning benefits and, consequently, incentives to exert effort (direct effect). On another hand, higher losing prizes eliminate the advantage of a stronger player and make the contest more balanced. As a result, the previously disadvantaged opponent can win with a higher probability and gets more stimuli to compete. If heterogeneity is severe, this positive equilibrium effect dominates the direct one. Thus, expected aggregate effort increases.

If contestants' preferences and / or skills are similar, a stronger participant has no significant advantage. In addition, higher losing rewards cause a comparable reduction in winning benefits of both players. Then, the positive equilibrium effect can never compensate losses in contestants' effort induced by cutting the prize spread. As a result, the "Winner–Takes–All" allocation supports the highest expected aggregate effort.

Notably, positive losing rewards could never benefit the designer in a single-item case. In this setting, contestants' types become aligned if and only if winning and losing prizes are identical. The allocation, however, gives no incentives to compete and is never optimal. On the contrary, in the two-dimensional setup it is possible to provide strictly positive winning benefits for both participants and at the same time make the contest even by increasing the losing prize. The result does not require convexity of the cost function as a traditional argument in favor of multiple positive rewards.[71] The key ingredient is asymmetry in contestants' valuations.

To highlight the importance of multi-dimensional prizes in job promotion and other similar settings, I structurally estimate the model using data from first-round matches of two professional tennis tournaments, the Australian Open and the French Open. Relevant prize dimensions are money and the ATP ranking points (career concerns). The analysis shows that both reward components shape players' incentives to compete. If one neglects the points (non-monetary) dimension, the model underestimates heterogeneity and overestimates the incentive effect of monetary prizes. This result highlights the direction to improve reduced-form tests of the contest theory that always overlook the multi-dimensionality aspect.

Counterfactual experiments illustrate the existence of alternative prize allocations that can improve aggregate effort. On average, the French Open (the Australian Open) would benefit from increasing (decreasing) monetary rewards for first-round losers. If the managers of both contests were allowed to change the ranking points allocation, they would prefer zero losing prizes in this good. The difference stems from the fact that the French Open tends to couple more heterogeneous players. This pattern can be explained with contest-specific matching policies.

Redistributing 5% of money and 2% of ranking points from final winning rewards to first-round losing prizes could improve expected aggregate effort in matches with strong heterogeneity by more than 4.9% (3.4%) for the French Open (the Australian Open). Compared to the "Winner–Takes–All" allocation, the gain would be 5.6% (3.7%) in the French Open (the Australian Open). On the contrary, relatively even matches never benefit from positive losing prizes. These findings go in line with the theory and can be used in the job promotion setting. For example, when managers must distribute monetary and non-monetary prizes between heterogeneous workers, they may gain if assign positive losing benefits. If competitors have similar skills and / or preferences, it is optimal to give everything to the winner.

To conclude, the paper highlights the importance of multi-dimensional incentives in asymmetric contests, both in theoretical and empirical terms. The presence of additional reward items helps the designer to mitigate the negative effect of strong heterogeneity on effort exertion, if the prize for a loser increases. The structural analysis has confirmed that at least two goods (namely,

[71] Though, even with the convex cost specification it would never be possible to support the optimality of non-zero benefits for the last loser under single-item rewards.

money and career concerns) shape the incentives of workers (professional athletes) in job promotion (professional tennis competitions). These findings indicate that multi-dimensional preferences can affect the contestants' behavior and, as a consequence, the optimal prize allocation significantly.

In the end, I indicate how this work could foster future research. First, one can introduce more than two players and prize items. This modified version would characterize a wider range of job promotion interactions and other contest-type settings. Another way to extend the framework is to look at multi-stage bilateral elimination contests with heterogeneous participants. In this case, the designer may want to maximize not only the total effort but also the winning probabilities of stronger players, and this results in a trade-off. To solve the problem, the designer can use two instruments: the prize allocation and the matching policy. The current empirical application highlights the importance of these two aspects to the contest setting.[72] Finally, in terms of structural estimation, it would be interesting to analyze female players whose preferences may differ from those of males and discover if there are any gender-specific patterns.

[72] At the earlier stages of professional tennis tournaments managers care not only about aggregate effort. They also want stronger players to be advanced in the contest with a higher probability. This is the reason why seeding policies exist.

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Tables and Figures

Figure 1.5.1: Empirical Distribution of Contestants' Ranking Points

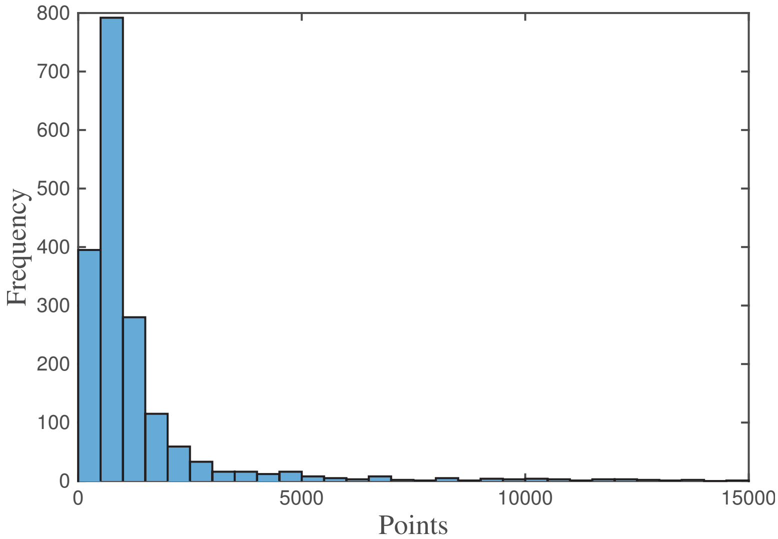


Table 1.6: The Australian Open Monetary Reward Schedules (A$1,000, CPI Adjusted), 2009–2015

Contest Stage	2015	2014	2013	2012	2011	2010	2009
1 Round	29.2	23.2	24.1	17.4	17.8	18.4	18.5
2 Round	50.8	38.7	39.7	28.9	28.4	29.7	29.5
3 Round	82.5	58.1	61.9	47.6	48.4	49.1	48.6
4 Round	148	104.4	109.1	95.1	82.6	83.9	83.9
1/4 Final	287.6	209	218.1	190.2	186.6	188.7	173.7
1/2 Final	549.8	418.1	436.2	380.5	373.2	377.4	347.9
Runner-Up	1,311.1	1,025.4	1,059.9	1,001.3	977.4	990.7	953.1
Winner	2,622.2	2,049.9	2,119.9	2,002.5	1,954.9	1,981.4	1,906.1
Total Prize Money	33,834.6	28,388.1	26,172.2	21,766.7	22,214.2	22,733.7	22,053.6
Prize Spread	131.9	97.9	101.8	91.6	88	89.3	85.3

Note: prize spreads are calculated as the difference between the continuation value of winning round 1 and the losing prize at that stage.

Table 1.7: Growth Rates in Monetary Rewards for the Australian Open (%), 2009–2015

Contest Stage	2015	2014	2013	2012	2011	2010
1 Round	25.64	–3.54	**38.28**	–2.01	–3.41	–.49
2 Round	31.10	–2.48	36.91	1.97	–4.33	.60
3 Round	42.03	–6.25	30.24	–1.79	–1.30	.94
4 Round	41.74	–4.23	14.65	15.11	–1.59	.13
1/4 Final	37.58	–4.15	14.65	1.95	–1.12	8.64
1/2 Final	31.51	–4.15	14.65	1.95	–1.12	8.50
Runner-Up	27.86	–3.26	5.86	2.44	–1.34	3.95
Winner	27.91	–3.30	**5.86**	2.44	–1.34	3.95
Total Prize Money	19.19	8.47	20.24	–2.01	–2.29	3.08
Prize Spread	22.92	5.12	**16.95**	–0.52	–1.84	3.62

Note: prize spreads are calculated as the difference between the continuation value of winning round 1 and the losing prize at that stage.

Table 1.9: Estimated Heterogeneity: Contest-Specific Effect

	Mean	Standard Deviation	Min	Max
The Australian Open				
Money	.109	.087	1E–11	.53
Points	.23	.69	4.2E–05	.7
The French Open				
Money	.096	.077	1E–11	.58
Points	.2	.64	1.7E–05	.77

Note: for every match j heterogeneity is measured as the absolute difference in contestants' skills and valuations:

$$het_j^A = \left|\tilde{\alpha}\left(x_{ij}\right) - \tilde{\alpha}\left(x_{-ij}\right)\right|, \, het_j^B = \left|\tilde{\beta}\left(x_{ij}\right) - \tilde{\beta}\left(x_{-ij}\right)\right|$$

where $\tilde{\alpha}(\cdot) = c(\cdot)\,\alpha(\cdot)$ and $\tilde{\beta}(\cdot) = c(\cdot)\,\beta(\cdot)$.

Table 1.8: Summary Statistics: Both Contests in 2009–2015

	N	Mean	Standard Deviation	Min	Max
Monetary Losing Prize (A$1,000)	1680	25.94	61.79	19.4	38.25
Monetary Winning Prize (A$1,000)	1680	2,171.5	396.55	1,605.38	3,100
Prize Spread (A$1,000)	1680	110.79	23.22	85.31	155.97
Total Prize Money (A$ 1,000)	1680	2.65E+04	4,752.56	2.18E+04	3.57E+04
Number of Unforced Errors per Set Played	1680	8.26	5.54	0	31
Age	1680	27.22	3.66	17	38
Body Mass Index (BMI)	1680	23.13	1.29	19.24	26.85
Ranking Points	1680	1249.29	1694.6	10	14,960
het_j^p	840	1342.54	2150.17	0	14,162
Betting Odd	1680	3.67	5.14	1	61
het_j^B	840	4.63	6.62	0	59.99

Dummy Variables

	N	Frequency of 1's
$Seed_i = 1$ if player i is seeded	1680	.25
$Hbias_i = 1$ if player i has a home bias	1680	.18
$Tour = 1$ for the Australian Open	1680	.5

Note: $het_j^p = |points_{ij} - points_{-ij}|$ ($het_j^B = |Bet_{ij} - Bet_{-ij}|$) approximates heterogeneity in match j as the absolute difference in contestants' ranking points (betting odds).

Figure 1.5.2: Goodness–of–Fit: Age- and Points-Based Groups & One-Dimensional Prizes

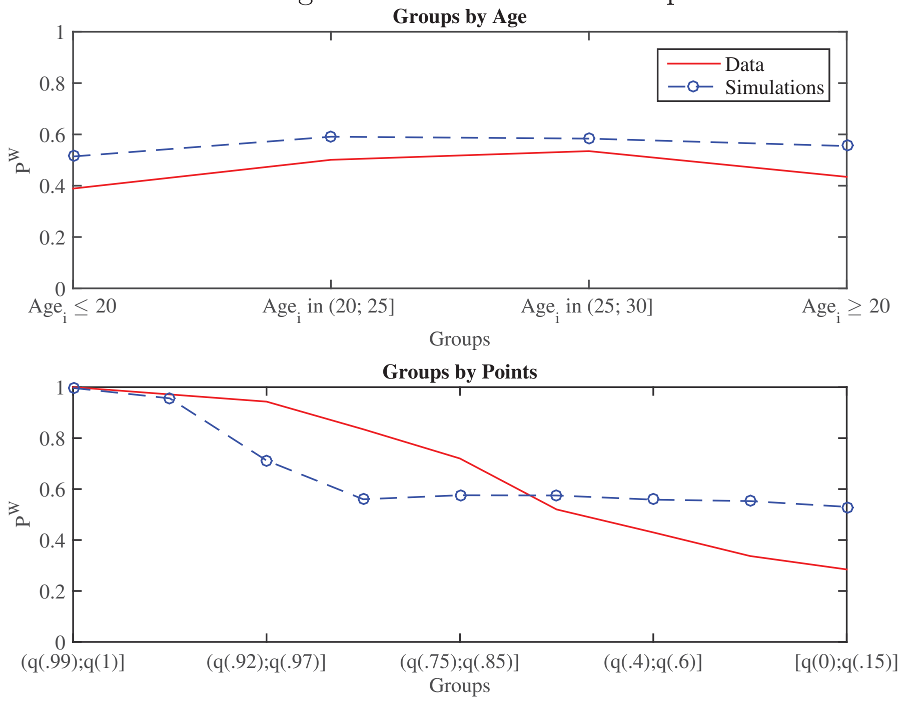


Table 1.10: Estimated Heterogeneity: No Contest Specific Effect

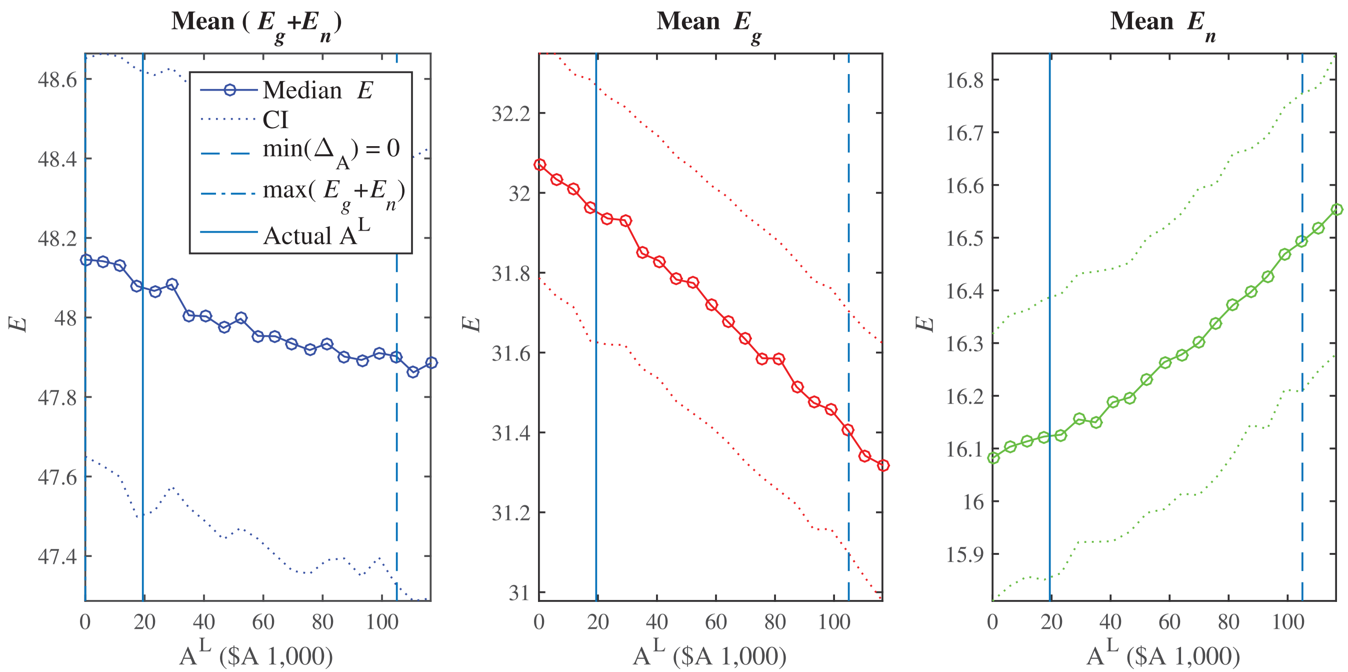
	Mean	Standard Deviation	Min	Max
The Australian Open				
Money	.091	.073	1.4E–11	.44
Points	.17	.53	1.7E–05	.58
The French Open				
Money	.096	.077	1E–11	.58
Points	.2	.64	1.7E–05	.77

Note: for every match j heterogeneity is measured as the absolute difference in contestants' skills and valuations:

$$het_j^A = \left|\tilde{\alpha}\left(x_{1j}\right) - \tilde{\alpha}\left(x_{2j}\right)\right|, \, het_j^B = \left|\tilde{\beta}\left(x_{1j}\right) - \tilde{\beta}\left(x_{2j}\right)\right|$$

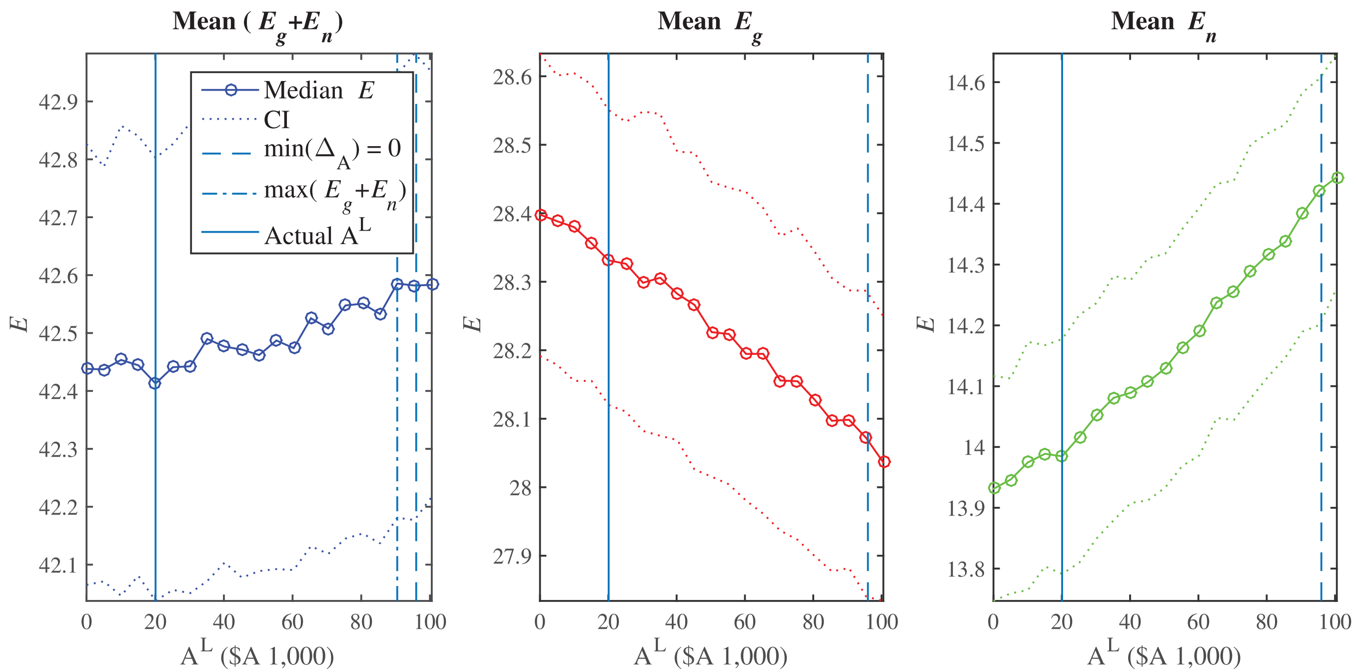
where $\tilde{\alpha}(\cdot) = c(\cdot)\,\alpha(\cdot)$ and $\tilde{\beta}(\cdot) = c(\cdot)\,\beta(\cdot)$.

Figure 1.5.3: Mean Expected Effort: the Australian Open and Fixed Prizes in Dimension B



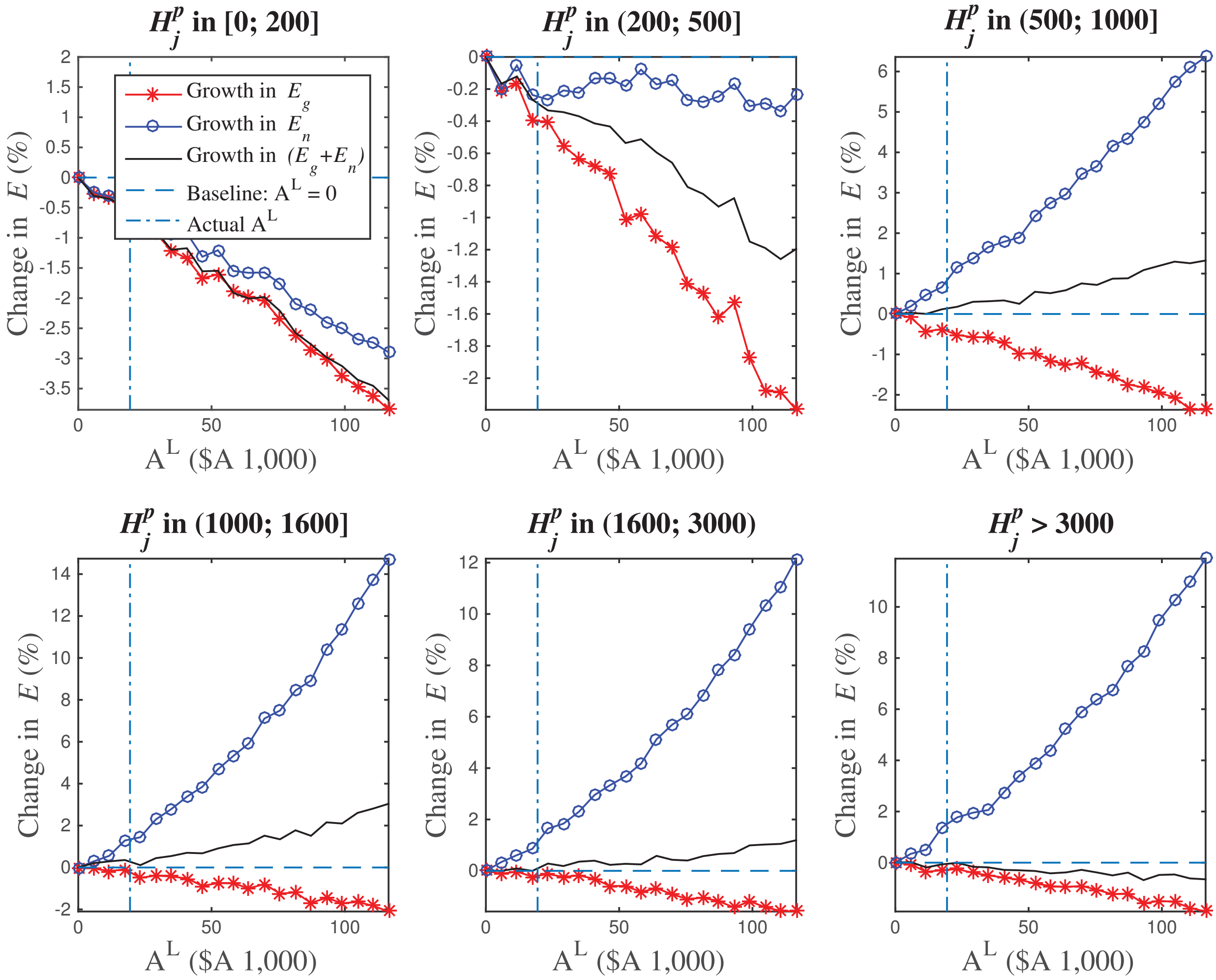
Note: $K = 821$; the total number of simulations is 1,000; solid lines with bubbles correspond to a 0.5–quantile of mean expected effort simulated distributions; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grid for monetary losing prizes is $\bar{A}^L = \left[0,\, \bar{u}A^L_a\right]$ with $\bar{u} = 6$ and 21 steps.

Figure 1.5.4: Mean Expected Effort: the French Open and Fixed Prizes in Dimension B



Note: $K = 821$; the total number of simulations is 1,000; solid lines with bubbles correspond to a 0.5–quantile of mean expected effort simulated distributions; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grid for monetary losing prizes is $\bar{A}^L = \left[0,\, \bar{u}A_a^L\right]$ with $\bar{u} = 5$ and 21 steps.

Figure 1.5.5: Group-Specific Improvement in Mean Expected Effort with respect to $A^L = 0$ (%): the Australian Open and Fixed Prizes in Dimension B



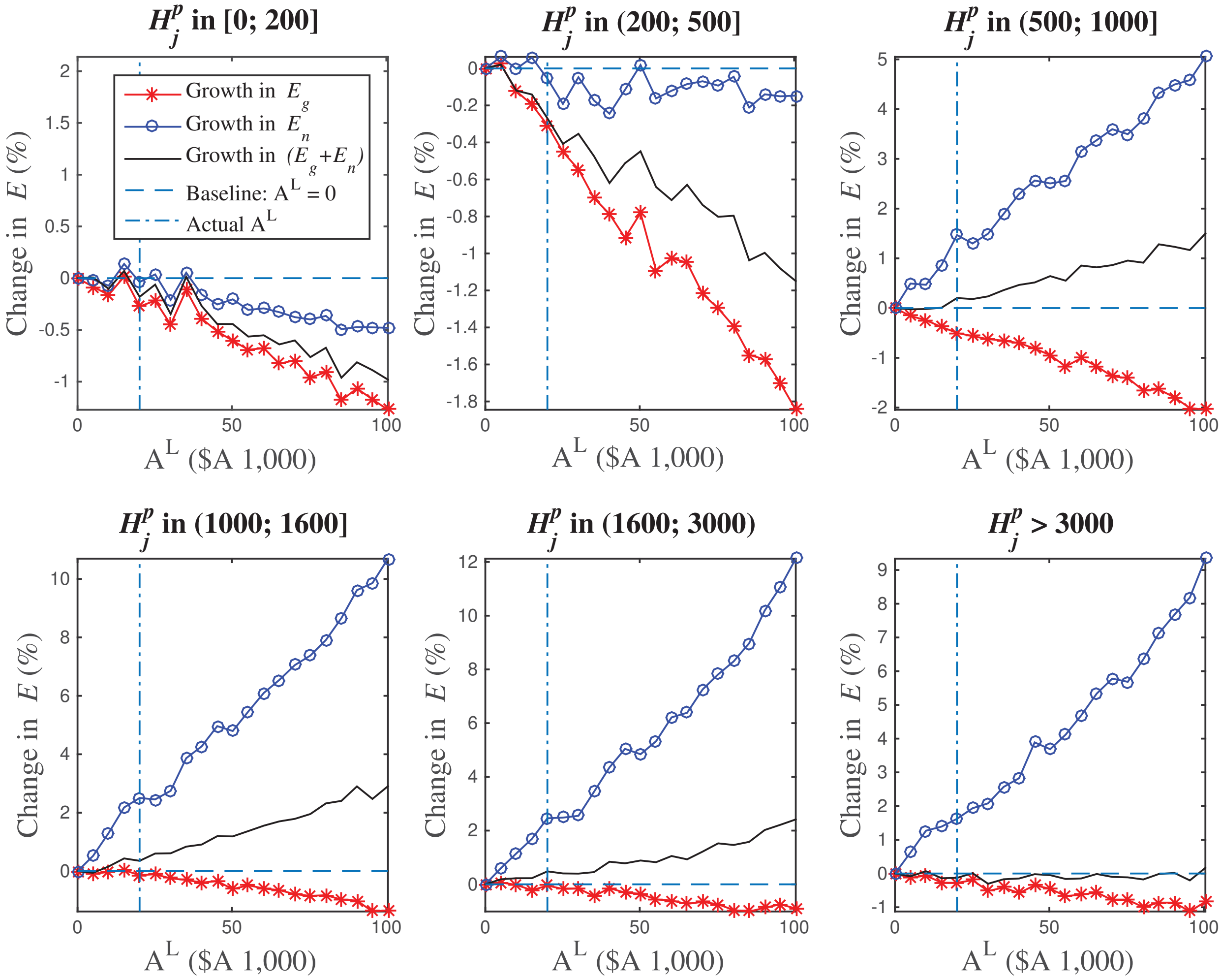
Note: $H_j^p = |points_{ij} - points_{-ij}|$ for every match j; $K = 821$; the total number of simulations is 1,000; solid lines with bubbles correspond to a 0.5–quantile of mean expected effort simulated distributions; the grid for monetary losing prizes is $\bar{A}^L = [0,\, \bar{u}A_a^L]$ with $\bar{u} = 6$ and 21 steps.

Table 1.11: Expected Aggregate Effort with One- and Two-Dimensional Rewards: Actual Prize Allocation

	The Australian Open	The French Open
No Prizes in Ranking Points	24.96	23.49
Two-Dimensional Prizes	48.02	42.12

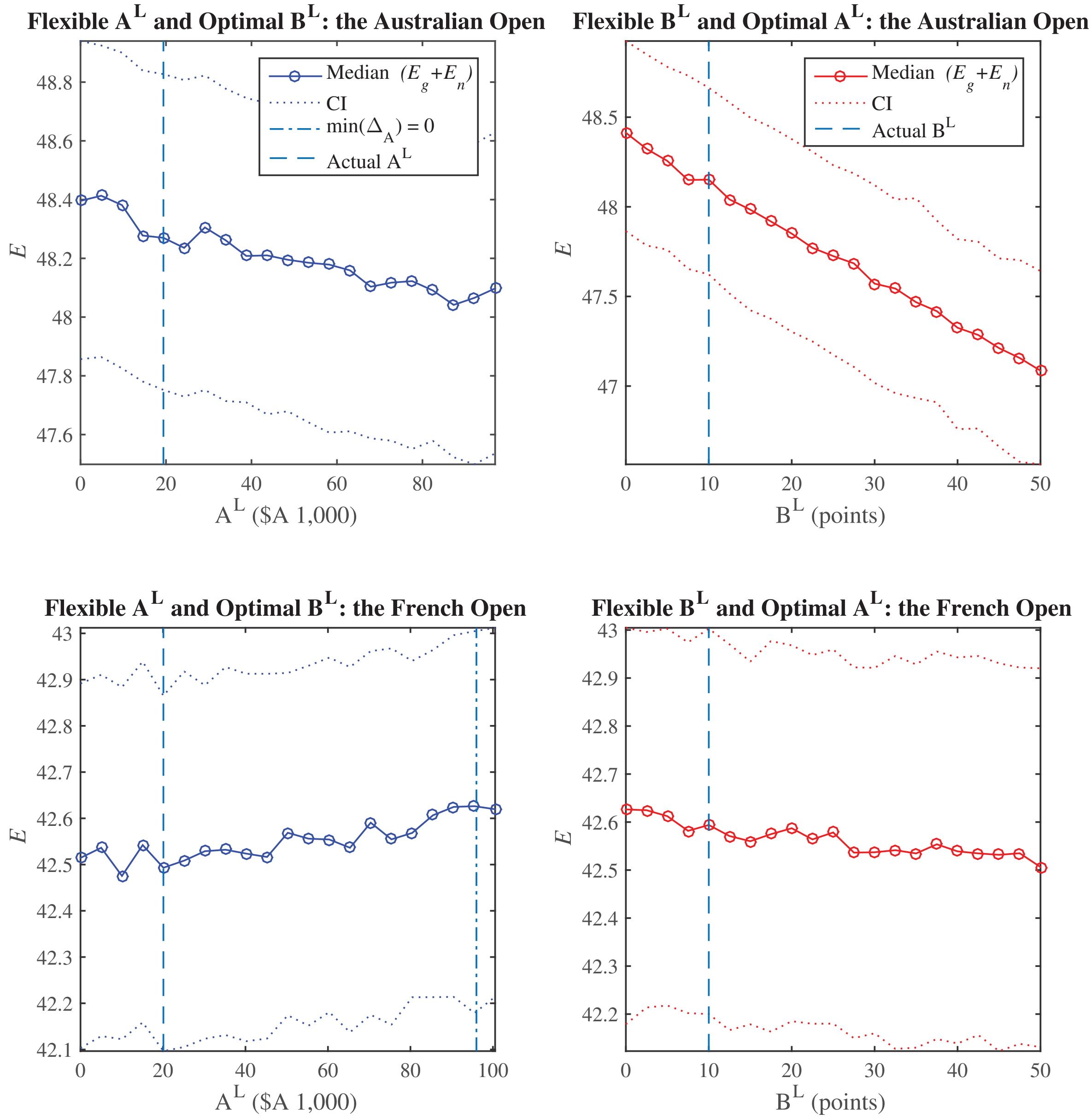
Note: total number of simulations is 1,000; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected aggregate effort simulated distribution.

Figure 1.5.6: Group-Specific Improvement in Mean Expected Effort with respect to $A^L = 0$ (%): the French Open and Fixed Prizes in Dimension B



Note: $H_j^p = |points_{ij} - points_{-ij}|$ for every match j; $K = 821$; the total number of simulations is 1,000; solid lines with bubbles correspond to a 0.5–quantile of mean expected effort simulated distributions; the grid for monetary losing prizes is $\bar{A}^L = [0,\, \bar{u}A_a^L]$ with $\bar{u} = 5$ and 21 steps.

Figure 1.5.7: Contest-Specific Mean Expected Aggregate Effort: Flexible Reward Schedules



Note: $K = 821$; the total number of simulations is 1,000; solid lines with bubbles correspond to a 0.5–quantile of mean expected effort simulated distributions; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grids for monetary and non-monetary losing prizes are $\bar{A}^L = [0, \bar{u}A_a^L]$ and $\bar{B}^L = [0, \bar{u}B_a^L]$ with $\bar{u} = 5$ and 21 steps.

Table 1.12: Best Group-Specific Prize Schedules: The Australian Open

Group	(A^L, B^L) **in The Best Prize Scheme**	**Highest Exp. Agg. Effort**	**Improvement in Exp. Agg. Effort w.r.t.** (0, 0)	**Improvement in Exp. Agg. Effort w.r.t.** (A_a^L, B_a^L)
$H_j^p \in [0, 200]$	(0, 0)	44.2	0%	1.2%
$H_j^p \in (200, 500]$	(0, 0)	44.8	0%	.8%
$H_j^p \in (500, 1000]$	(116.4, 3)	39.6	**1.1%**	.9%
$H_j^p \in (1000, 1600]$	(116.4, 42)	37.3	**3.7%**	3.4%
$H_j^p \in (1600, 3000]$	(116.4, 60)	36.9	**1.9%**	1.8%
$H_j^p > 3000$	(0, 0)	34.8	0%	.5%

Note: $H_j^p = |points_{ij} - points_{-ij}|$ for every match j; A^L is measured in A\$1,000; B^L is measured in points; total number of simulations is 1,000; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grids for monetary and non-monetary losing prizes are $\bar{A}^L = \left[0,\, \bar{u}A_a^L\right]$ and $\bar{B}^L = \left[0,\, \bar{u}B_a^L\right]$ with $\bar{u} = 6$ and 21 steps.

Table 1.13: Best Group-Specific Prize Schedules: The French Open

Group	(A^L, B^L) **in The Best Prize Scheme**	**Highest Exp. Agg. Effort**	**Improvement in Exp. Agg. Effort w.r.t.** (0, 0)	**Improvement in Exp. Agg. Effort w.r.t.** (A_a^L, B_a^L)
$H_j^p \in [0, 200]$	(0, 0)	45.1	0%	.5%
$H_j^p \in (200, 500]$	(0, 0)	45.2	0%	.6%
$H_j^p \in (500, 1000]$	(100.5, 32.5)	41.2	**1.8%**	1.6%
$H_j^p \in (1000, 1600]$	(100.5, 32.5)	39.1	**5.3%**	4.8%
$H_j^p \in (1600, 3000]$	(100.5, 50)	37.4	**5.6%**	4.9%
$H_j^p > 3000$	(100.5, 47.5)	35.1	.4%	.5%

Note: $H_j^p = |points_{ij} - points_{-ij}|$ for every match j; A^L is measured in A\$1,000; B^L is measured in points; total number of simulations is 1,000; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grids for monetary and non-monetary losing prizes are $\bar{A}^L = \left[0,\, \bar{u}A_a^L\right]$ and $\bar{B}^L = \left[0,\, \bar{u}B_a^L\right]$ with $\bar{u} = 5$ and 21 steps.

Appendix A. Theoretical Model: Extensions

Bundling vs. Two Simultaneous Contests

Suppose there exists an alternative way to use both goods. The designer can run two separate contests with one-dimensional rewards instead of a single competition with bundled prizes. If expected aggregate effort taken over former contests always exceeds the one induced in the latter case, results of Theorem 1.1 break down easily.

Assume the designer has two possibilities:[73]

1. Scheme b: run one contest, allocate prize bundles and induce expected aggregate effort $J^b\left(A^W,\, A^L,\, B^W,\, B^L\right)$.

2. Scheme ub: run two separate competitions with single-item prizes and take a sum of expected aggregate effort over them:
$$J^{ub}\left(A^W,\, A^L,\, B^W,\, B^L\right) = J\left(A^W,\, A^L,\, 0,\, 0\right) + J\left(0,\, 0,\, B^W,\, B^L\right)$$
where function $J\left(\cdot\right)$ was defined in the original model.

Let $W^b = \begin{pmatrix} A_b^W \\ B_b^W \end{pmatrix}$ and $L^b = \begin{pmatrix} A_b^L \\ B_b^L \end{pmatrix}$ ($W^{ub} = \begin{pmatrix} A_{ub}^W \\ B_{ub}^W \end{pmatrix}$ and $L^{ub} = \begin{pmatrix} A_{ub}^L \\ B_{ub}^L \end{pmatrix}$) be optimal prizes for scheme b (ub). In a single-item contest, the designer always assigns the highest possible prize spread (see the proof of Proposition 1.2, ***Lemma 5***):
$$W^{ub} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\, L^{ub} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
Theorem 1.1 characterizes optimal reward schedules for scheme b. Abusing notations, I introduce $J^l\left(W^l,\, L^l\right) \equiv J^l\left(A_l^W,\, A_l^L,\, B_l^W,\, B_l^L\right)$, $l = \{b,\, ub\}$. When optimal prize schedules are known, the designer faces the following problem:
$$\max\left\{J^b\left(W^b,\, L^b\right),\, J^{ub}\left(W^{ub},\, L^{ub}\right)\right\}$$
As before, R defines a set of all feasible valuation profiles:
$$R = \{r:\, \alpha_i \geq 0,\, \beta_i \geq 0,\, i = \{g,\, n\}\}$$
Theorem 1.1 states the existence of valuation profiles such that scheme b (with or without positive losing prizes) induces higher expected aggregate effort:

[73] Labels b and ub denote bundled and unbundled prize schedules, respectively.

Theorem 1.1. *There exists a non-empty subset of* R*,* R^b*, such that for any* $\{\alpha_g, \alpha_n, \beta_g, \beta_n\} \in R^b$ *the designer prefers one contest with bundled prizes to two separate contests with unbundled prizes.*

Proof. See Appendix E. □

Set R^b contains two types of valuation profiles. The first type supports optimal bundles where a winner takes all. These profiles must be such that the greediest player values only one good more than the opponent ($\alpha_g > \alpha_n$, $\beta_g < \beta_n$ or vise versa). The result has the following interpretation. Suppose player g likes both items more ($\alpha_g > \alpha_n$ and $\beta_g > \beta_n$). The "Winner–Takes–All" bundle is optimal when contestants have relatively homogeneous preferences over both dimensions. Since player g values goods more, his relative power is higher in the competition with bundled prizes than in two separate contests. As a result, the opponent has less incentives to exert effort in the former case. Formally, for $\alpha_g > \alpha_n$, $\beta_g > \beta_n$ player n chooses $e > 0$ with a lower probability when the designer runs a single contest and allocates the "Winner–Takes–All" bundle:[74]

$$P\left(e_n > 0 |\, b,\, WTA\right) < P\left(e_n > 0 |\, ub\right)\; if\; \alpha_g > \alpha_n,\, \beta_g > \beta_n$$

If contestant g likes only one good more, the inequality can change its sign. Then, player n exerts strictly positive effort with a higher probability when the designer allocates the "Winner–Takes–All" bundle. Given that participants have relatively homogeneous preferences, two separate competitions with one-dimensional prizes induce sufficient expected aggregate effort. Thus, the designer prefers scheme ub over b when player g values both goods more. Otherwise, he can run one contest and allocate the "Winner–Takes–All" bundle.

Another type of valuation profiles entering R^b supports optimal bundles with positive losing prizes. A sufficient condition to make this reward schedule optimal is strong heterogeneity in players' preferences. This also means that single-item contests are very uneven and result in low expected aggregate effort. Two-dimensional prizes with positive losing benefits help the designer to mitigate a negative effect players' heterogeneity has on effort exertion.[75] To illustrate this point, I provide a numerical example.

[74]I use Proposition 1.1 to calculate these probabilities:

$$P\left(e_n > 0 |\, b,\, WTA\right) = \frac{\alpha_n+\beta_n}{\alpha_g+\beta_g}$$
$$P\left(e_n > 0 |\, ub\right) = \frac{\alpha_n\beta_n}{\alpha_g\beta_g} + \frac{\alpha_n}{\alpha_g}\left(1-\frac{\beta_n}{\beta_g}\right) + \frac{\beta_n}{\beta_g}\left(1-\frac{\alpha_n}{\alpha_g}\right)$$

where the latter is a probability that player n exerts positive effort at least in one contest with single-item prizes. Then:

$$P\left(e_n > 0 |\, b,\, WTA\right) < P\left(e_n > 0 |\, ub\right) \Leftrightarrow -\alpha_n\beta_g\left(\beta_g-\beta_n\right) - \beta_n\alpha_g\left(\alpha_g-\alpha_n\right) < 0$$

and the inequality always holds for $\alpha_g > \alpha_n$, $\beta_g > \beta_n$.

[75]For more details on this mechanism, see Subsection 1.3.3.1.

Example. Assume $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$. I start from scheme b. The "Winner–Takes–All" schedule results in $J^b(1,\, 0,\, 1,\, 0) = 7.8$. Player 1 is the "greediest-to-win" ($g = 1$). For given α_g and α_n, it is always optimal to assign the highest prize spread in good A (see the proof of Theorem 1.1). Type g is more sensitive to incentives in dimension B ($\frac{\alpha_g}{\beta_g} < \frac{\alpha_n}{\beta_n}$). Then, positive losing benefits can appear only in dimension B. In fact, the optimal allocation of item B for specified preferences is $B^W = \frac{3}{7}$, $B^L = \frac{4}{7}$ (see the proof of Theorem 1.1). This alternative prize schedule induces $J^b(1,\, 0,\, \frac{3}{7},\, \frac{4}{7}) = 8.9$ and dominates the "Winner–Takes–All" scheme. If the designer runs two separate single-item contests, he gets $J^{ub}(1,\, 0,\, 1,\, 0) = 9.1$. Thus, scheme ub must be chosen. This is the case because heterogeneity in dimension B is not strong enough, and the corresponding contest with single-item prizes generates enough effort to make scheme ub beneficial.

Next, take the same $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ combined with $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 0.1 \end{pmatrix}$. Now the difference between β_1 and β_2 increases. Again, player 1 is of type g, and his incentives show relatively more sensitivity in dimension B ($\frac{\alpha_g}{\beta_g} < \frac{\alpha_n}{\beta_n}$). Consider scheme b. The "Winner–Takes–All" allocation induces $J^b(1,\, 0,\, 1,\, 0) = 6.9$. The optimal prize schedule for given preferences requires $B^W \approx \frac{11}{25}$, $B^L \approx \frac{14}{25}$, and expected aggregate effort is $J^b(1,\, 0,\, \frac{11}{25},\, \frac{14}{25}) \approx 9$. When the designer implements scheme ub, he gets $J^{ub}(1,\, 0,\, 1,\, 0) = 8.6$. As a result, when heterogeneity in dimension B is very strong, the designer does not benefit from two contests with single-item rewards and prefers to create a bundle with a positive losing prize.

Overall, Theorem 1.1 strengthens the results of the original model: bundling with positive losing prizes can be beneficial even if the designer has a freedom to run separate contests over two dimensions.

Asymmetric Information about Contestants' Types

In this section, I introduce asymmetric information about contestants' types. This extension is important for two reasons. First, players often have unobservable characteristics that can affect their effort choices. Second, in many real-life contests the designer must commit to the prize allocation before he learns exact matching. For example, contest organizers in professional tennis announce monetary reward schedules before the final draw is known.

Suppose all key assumptions of the original model hold. However, now α_i is private information of player i:

$$\alpha_i = \{\underline{\alpha}_i,\, \bar{\alpha}_i\},\ 0 < \underline{\alpha}_i < \bar{\alpha}_i,\ i = \{1,\, 2\}$$

where realizations of α_1 and α_2 are independent, $P(\alpha_i = \bar{\alpha}_i) = k \ \forall\ i = \{1,\, 2\}$. For tractability reasons, β_1 and β_2 are assumed to be common knowledge. This allows me to restrict a number of types every contestant has by two and avoid excessive parametrization.[76] Without loss of generality, I fix $\beta_1 > \beta_2$.

Define $\bar{U}_i^k = \bar{\alpha}_i A^k + \beta_i B^k$ and $\underline{U}_i^k = \underline{\alpha}_i A^k + \beta_i B^k$ if player i wins prize k. As every contestant has two types, I introduce the following definition:

Definition 1.4. *Let $\bar{t}_i(\triangle_A, \triangle_B) = \bar{U}_i^W - \bar{U}_i^L \equiv \bar{t}_i$ and $\underline{t}_i(\triangle_A, \triangle_B) = \underline{U}_i^W - \underline{U}_i^L \equiv \underline{t}_i$ be winning benefits of contestant i with realized valuations $\bar{\alpha}_i$ and $\underline{\alpha}_i$, respectively. Define $t_i^g = \max\{\bar{t}_i, \underline{t}_i\}$ and $t_i^n = \min\{\bar{t}_i, \underline{t}_i\}$. Then $\bar{t}_i$ is the **"greediest-to-win" type of contestant** i ($g = i$) if and only if $\bar{t}_i = t_i^g$; otherwise, $\bar{t}_i$ is not the **"greediest-to-win" type of contestant** i ($n = i$).*

I do not compare types between contestants but identify two states the same player can face. As before, every participant chooses his effort taking into account announced prizes and the competitor's action. When I analyze the game between contestants, I look at a particular class of equilibria.

Definition 1.5. *Let $e^*\left(t_i^j\right)$ be effort type j of contestant i exerts in equilibrium. The equilibrium is in monotonically increasing strategies if and only if $e^*(t_i^g) \geq e^*(t_i^n)$ for any $i = \{1, 2\}$.*

The definition says that in equilibrium g-types associated with higher winning benefits must never exert less effort than n-types. Also, I require non-triviality of equilibria:

Definition 1.6. *The equilibrium is trivial if and only if at least one g-type chooses $e = 0$ with probability 1. Otherwise, the equilibrium is non-trivial.*

Let $G_i\left(t_i^j, e\right)$ be a probability that type j of contestant i chooses at most e, and s_i^j denotes a support of $G_i\left(t_i^j, e\right)$. Proposition 1.4 characterizes contestants' equilibrium behavior:

Proposition 1.4. *For $\min\{t_1^g, t_2^g\} > 0$ there exists a unique non-trivial equilibrium in monotonically increasing strategies such that:*

- *At least one type places an atom at zero: $\exists\, i = \{1, 2\}, j = \{g, n\} : G_i\left(t_i^j, 0\right) > 0$;*
- *There is no $e > 0$ played with a positive probability;*

[76] Equivalently, one could assume private information about β_i but state that two random variables are perfectly correlated:

$$\beta_i = \left\{\underline{\beta}_i, \bar{\beta}_i\right\}, 0 < \underline{\beta}_i < \bar{\beta}_i$$
$$P\left(\beta_i = \bar{\beta}_i \mid \bar{\alpha}_i\right) = 1,\ P\left(\beta_i = \underline{\beta}_i \mid \underline{\alpha}_i\right) = 1$$
$$P\left(\beta_i = \bar{\beta}_i\right) = P\left(\alpha_i = \bar{\alpha}_i\right) = k\ \forall\, i = \{1, 2\}$$

- *Supports of* $G_i\left(t_i^g,\, e\right)$*,* $i = \{1,\, 2\}$ *have the same supremum:* $\sup\left(s_1^g\right) = \sup\left(s_2^g\right)$*;*
- *Supports of* $G_i\left(t_i^n,\, e\right)$*,* $i = \{1,\, 2\}$ *have the same infimum, and it is equal to zero:* $\inf\left(s_1^n\right) = \inf\left(s_2^n\right) = 0$.

Proof. See Appendix E. □

To construct this equilibrium, I use an algorithm developed by Siegel (2014) for asymmetric all–pay auctions with two participants, private information and discrete types. I extend this characterization to the case of perfectly divisible multiple goods and introduce prize-dependent types.

When the second strongest type has a non-positive winning benefit ($\min\{t_1^g,\, t_2^g\} \leq 0$), the equilibrium becomes trivial.[77] Otherwise, g-types of both contestants choose $e > 0$ with a strictly positive probability. As before, the equilibrium features mixed strategies. All effort choices must belong to $s = [0,\, \min\{t_1^g,\, t_2^g\}]$. The equilibrium partitions s into intervals of different length. On every interval, particular types of contestants 1 and 2 compete by randomizing uniformly. The equilibrium partition depends on the prize structure (W and L), the probability distribution (k) and includes up to 3 intervals. One can find the greediest competitor for every element of the equilibrium partition:

Definition 1.7. *Define order* Q *over elements of the equilibrium partition where* $q = 1$ *corresponds to* $e = 0$*. Let* $t_{i,\,q}$ *be a type of contestant* i *playing on interval* $q \in Q$*. Then,* $t_{i,\,q}$ *is the "greediest-to-win" on interval* $q \in Q$ *(*$g_q = i$*) if and only if* $t_{i,\,q} = \max\{t_{1,\,q},\, t_{2,\,q}\}$*.*

Using these notations, I define a relative power of type g_q in this setting:

Definition 1.8. *A relative power of contestant* g_q *on interval* $q \in Q$*,* $q > 1$ *is a probability that his competitor plays on interval* $q - 1 \in Q$*.*

This notion of the relative power is very similar to one introduced for the case of symmetric information. In the lowest interval of the equilibrium partition ($q = 2$), the relative power of player g_q is just a probability that the opponent chooses $e = 0$. As in the original model, two-dimensional prizes allow the designer to affect the equilibrium partition by changing winning and losing prizes.[78] The mechanism driving this result was described in Subsection 1.3.2.

The designer chooses the prize allocation that maximizes expected aggregate effort, $J_k(\cdot)$, given feasibility constraints. Let r_k be a probability–valuation profile, and R_k denotes a set of all feasible r_k's:

$$R_k = \{r_k :\ \underline{\alpha}_i \geq 0,\ \bar{\alpha}_i \geq 0,\ \beta_i \geq 0,\ k \in [0,\, 1]\}$$

[77] The argument is exactly the same as the one provided for the case of $t_n \leq 0$ in Proposition 1.1.

[78] In case of single-item rewards, this value is constant.

To simplify the analysis, I fix two patterns in players' preferences. First, I assume that contestant 1 has stronger willingness to win for both realizations of α_1 and α_2 when the "Winner–Takes–All" schedule is implemented:

$$t_1^j\,(1,\,1) > t_2^j\,(1,\,1)\ \forall\, j = \{g,\, n\} \Leftrightarrow \underline{\alpha}_1 > \bar{\alpha}_2 + \beta_1 - \beta_2$$

Second, I look at symmetric profiles:

Definition 1.9. *Probability–valuation profile r_k is symmetric if and only if* $\min\{\underline{\alpha}_1,\,\underline{\alpha}_2\} > \beta_1$ *or* $\max\{\bar{\alpha}_1,\,\bar{\alpha}_2\} < \beta_2,\ \beta_1 > \beta_2$. *Otherwise, probability–valuation profile is asymmetric. $R_k^s \subset R_k$ is a set of all symmetric probability–valuation profiles.*

With symmetric profiles, both contestants enjoy the same good more for all realizations of α_1 and α_2. Proposition 1.5 states the existence of probability–valuation structures that satisfy all imposed constraints and make positive losing prizes optimal:[79]

Proposition 1.5. *For $\beta_1 > \beta_2$ and $\underline{\alpha}_1 > \bar{\alpha}_2 + \beta_1 - \beta_2$ there exists a non-empty subset of R_k^s, $R_k^{L,\,s}$, such that for any probability–valuation profile in $R_k^{L,\,s}$ the designer uses both goods completely and assigns a positive losing prize in dimension A.*

Proof. See Appendix E. □

Mechanisms driving the optimality of positive losing prizes are exactly the same as described in Subsection 1.3.3.1. Higher losing rewards lower winning benefits and, consequently, incentives to exert effort (direct effect). At the same time, it reduces the advantage of the greediest player on every interval of the equilibrium partition and, consequently, incentivizes the opponent to compete more (equilibrium effect). When contestants' preferences are very heterogeneous, the latter effect dominates, and expected aggregate effort goes up.

Overall, key results of the baseline model hold even when asymmetric information is introduced, and it makes the proposed framework robust in this respect.

[79] Since the result concerning the optimality of positive losing prizes is the most striking one, I concentrate on proving only this case and show that similar mechanisms work under asymmetric information. I do not analyze the optimality of the "Winner–Takes–All" bundle separately although there exist probability–valuation profiles supporting it.

Appendix B. The Effort Proxy Performance in the Reduced-Form Tests

This section shows that the proposed measure based on a number of unforced errors per set played can approximate contestants' effort well. I use the unbalanced panel of 320 players participated either in the Australian Open or the French Open in 2009–2015. Following the discussion in Section 1.4, I do not address selection issues here.

Since skills and preferences cannot be recovered in the reduced-form setting, one must approximate players' heterogeneity. Following methodology developed in other empirical tests of the contest theory, I measure heterogeneity by taking the absolute difference in ranking points of the competitors (variable het_j^P for game j). [80]

Let A and B denote money and ranking points (career concerns), respectively. I characterize first-round winning prizes as a continuation value of being advanced in the contest. Suppose success and failure at later stages of the tournament are equally probable. Also, assume identities of potential future opponents do not matter.[81] With this approach, prize spreads are the same for all players. As a result, heterogeneity in first-round games stems only from different skills and asymmetric preferences.

Following previous reduced-form tests of the contest theory, in this section I consider only monetary incentives. Standard tournament models predict that good effort proxies must reflect the following patterns:

1. Effort increases in the prize spread ($\triangle_A = A^W - A^L$) (Hypothesis 1, or H_1);

2. Effort decreases in contestants' heterogeneity (het_j^p) (Hypothesis 2, or H_2).

In addition, I want to see how monetary losing prizes affect effort choices. However, with the player-invariant spread specification this is not possible because the two values are strongly correlated ($corr\left(A^L, \triangle_A\right) = 0.92$). Table 1.15 contains estimation results. H_1 and H_2 cannot be rejected in all specifications: individual effort increases (decreases) when monetary prize spreads (heterogeneity) grow. In this respect, the proposed measure behaves as a good proxy for players' effort.

To capture the difference in contestants' preferences, I introduce another specification of the prize spread. Players are divided into six groups based on their ranking. For each of them, I calculate empirical frequencies of being advanced to a particular contest level (see Table 1.14).

[80]For instance, see Sunde (2009).

[81]This strategy was used in several empirical papers (for instance, see Silverman and Seidel (2011), Ivankovic (2007)) with the following argument: winning and losing probabilities of different players average to "1/2–1/2".

The algorithm generates non-parametric group-specific continuation values and prize spreads ($\triangle_A^G$). Now, monetary losing benefits can also be used as an explanatory variable ($corr\left(A^L,\, \triangle_A^G\right) = 0.07$).

When I analyze all matches together, H_1 cannot be rejected (see Table 1.16, specifications (1)–(3)), and the candidate effort measure performs well. However, there are some interesting features to be emphasized. First, monetary losing prizes matter for contestants' effort exertion, and the effect is positive. Second, there exists the group of participants whose effort decreases in prize spreads but grows in losing benefits (see specification (4) in Table 1.16). H_2 finds support in all specifications.

Finally, one can argue more unforced errors stem from risk-taking behavior but not lower effort.[82] To address this point, I suggest the following approach. I assume that contestants taking more risk hit the ball stronger. As a result, their serving speed increases. Then if the number of unforced errors per set also includes risk-taking, it must be positively correlated with the latter variable. Table 1.17 shows that relevant regression coefficients are insignificant. As a result, the proposed proxy does not reveal evidence of risk-taking.

Table 1.14: Round–Specific Empirical Frequencies of Winning for Different Contestants' Groups

	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$
Round 2	1	1	0.95	0.62	0.32	0.14
Round 3	1	0.93	0.92	0.41	0.17	0.38
Round 4	1	0.85	0.68	0.22	0.2	0
1/4 *Final*	0.73	0.73	0.39	0.33	0	0
1/2 *Final*	0.88	0.5	0.33	0	0	0
Final	0.71	0.25	0.33	0	0	0

Note:

1. Cells of the table contain empirical frequencies of group l (column) winning round j (row) in the Australian Open or the French Open in 2009–2015, $i = \{1, ..., 6\}$ $j = \{2, 3, 4, 1/4\,final, 1/2\,final, final\}$.

2. Groups based on ranking points look as follows:

$$i \in l \Leftrightarrow points_i \in [G_{7-l},\, G_{7-l+1})\,,\; l = \{1, ..., 6\}$$
$$G_k \in G,\; G = \{0, 500, 1000, 3000, 6000, 10000, \max\left(points_i\right)\}$$

3. One can calculate group–specific prize spreads using empirical frequencies of winning and reward schedules.

[82]Since the theoretical setup assumes risk-neutrality, I prefer to isolate this channel.

Table 1.15: Individual Effort: Fixed–Effect Specification and Constant Prize Spreads

	(1) No Other Controls	(2) Game and Tournament Controls–I	(3) Game and Tournament Controls–II
$\triangle_{A,\,t}$, A$1000	0.048*** (0.017)	0.052*** (0.017)	0.05*** (0.017)
het^p_j	–0.4E–03*** (0.7E–0.4)	–0.5E–03*** (0.8E–0.4)	–0.5E–03*** (0.8E–0.4)
$\frac{A^L_t}{\triangle_{A,\,t}}$	No	No	–5.21 (15.14)
Match-Spec. Controls	No	Yes	Yes
Contest-Spec. Controls	No	Yes	Yes
Time Effects	Yes	Yes	Yes
Contest-Spec. Trend	Yes	Yes	Yes
	$N = 1647$ $F\,(9,\,532) =$ 8.42	$N = 1642$ $F\,(14,\,531) =$ 7.33	$N = 1642$ $F\,(15,\,531) =$ 6.84

Note: dependent variable – individual effort measured as $\tilde{e}_{ij} = \hat{u} - \tilde{u}_{ij}$ where $\tilde{u}_{ij}$ is a number of unforced errors per set player i made in match j, $\hat{u} = \max_{i,\,j} \tilde{u}_{ij}$; standard errors are clustered over players; *, **, and *** correspond to 90%, 95%, and 99% significance; other controls include seeding policies of the contests, players' and their competitors' individual characteristics, betting odds, growth rates of A^L_t and $\triangle_{A,\,t}$, total prize money.

Table 1.16: Individual Effort: Fixed–Effect Specification and Flexible Prize Spreads

	(1) No Other Controls	(2) Game and Tournament Controls–I	(3) Game and Tournament Controls–II	(4) Restricted Set of Players
$\triangle^G_{A,\,t}$, A$1000	0.004*** (0.001)	0.0035** (0.001)	0.0036** (0.001)	–0.24** (0.001)
het^p_j	–0.5E–03*** (0.8E–0.4)	–0.5E–03*** (0.8E–0.4)	–0.5E–03*** (0.9E–0.4)	–0.45E–03*** (0.13E–0.4)
A^L_t	No	0.18** (0.083)	0.19** (0.08)	0.716*** (0.023)
Match-Spec. Controls	No	No	Yes	Yes
Contest-Spec. Controls	No	No	Yes	Yes
Time Effects	Yes	Yes	Yes	Yes
Contest-Spec. Trend	Yes	Yes	Yes	Yes
	$N = 1647$ $F\,(9,\,532) =$ 7.66	$N = 1647$ $F\,(10,\,532) =$ 7.53	$N = 1642$ $F\,(14,\,531) =$ 6.45	$N = 649$ $F\,(14,\,397) =$ 3.99

Note: dependent variable – individual effort measured as $\tilde{e}_{ij} = \hat{u} - \tilde{u}_{ij}$ where $\tilde{u}_{ij}$ is a number of unforced errors per set player i made in match j, $\hat{u} = \max_{i,\,j} \tilde{u}_{ij}$; standard errors are clustered over players; *, **, and *** correspond to 90%, 95%, and 99% significance; other controls include seeding policies of the contests, players' and their competitors' individual characteristics, betting odds, growth rates of A^L_t and $\triangle^G_{A,\,t}$, total prize money; specification (4) includes only contestants holding at most 1,000 ranking points and playing in matches with $het^p_j > 300$.

Table 1.17: Number of Unforced Errors and Risk-Taking Behavior

	Model I	Model II	Model III
Fastest Serve Speed, km/h	0.03 (0.04)	No	No
Average 1st Serve Speed, km/h	No	–0.9E–03 (0.035)	No
Average 2nd Serve Speed, km/h	No	No	0.02 (0.03)
	$N = 813$ $F\,(12,\,367) =$ 3.3	$N = 813$ $F\,(12,\,367) =$ 3.4	$N = 813$ $F\,(12,\,367) =$ 3.39

Note: dependent variable – number of unforced errors a player made per set; standard errors are clustered over players; *, **, and *** correspond to 90%, 95%, and 99% significance; other controls include time controls, contest-specific trends, seeding policies, players' and their competitors' individual characteristics.

Appendix C. Structural Modeling: Assumptions on the Player-Specific Noise Distribution

In the structural setup, I assume that contestants get additional individual-specific utility (or disutility) when lose:

$$U_i^L\left(A^L,\, B^L,\, \varepsilon_i\right) = c_i\left(\alpha_i A^L + \beta_i B^L\right) + \varepsilon_i,\, \varepsilon_i \sim iidF_i\left(\varepsilon\right),\, \varepsilon \in \left[\underline{\varepsilon}_i,\, \bar{\varepsilon}_i\right]$$

where $F_i\left(\varepsilon\right)$ is a truncation of a mean zero normal distribution with standard deviation σ_ε:

$$f_i\left(\varepsilon\right) = \frac{\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\bar{\varepsilon}_i}{\sigma_\varepsilon}\right) - \Phi\left(\frac{\underline{\varepsilon}_i}{\sigma_\varepsilon}\right)},\, \varepsilon \in \left[\underline{\varepsilon}_i,\, \bar{\varepsilon}_i\right]$$

With this approach, contestants' types become random:

$$\tilde{t}_i = c_i\alpha_i\triangle_A + c_i\beta_i\triangle_B - \varepsilon_i = t_i - \varepsilon_i$$

To match the original theoretical model, I must restrict the support of ε_i in the following way:

Assumption 1: Losing never results in negative utility:

$$U_i^L\left(A^L,\, B^L,\, \varepsilon_i\right) = c_i\left(\alpha_i A^L + \beta_i B^L\right) + \varepsilon_i \geq 0 \Leftrightarrow \underline{\varepsilon}_i = -c_i\left(\alpha_i A^L + \beta_i B^L\right)$$

Further, recall the definition of contestants' types. Player i is the "greediest-to-win" ($\tilde{t}_i = \tilde{t}^g$) if and only if $\tilde{t}_i > \tilde{t}_{-i}$:

$$\tilde{t}_i > \tilde{t}_{-i} \Leftrightarrow \varepsilon_i - \varepsilon_{-i} < t_i - t_{-i}$$

Now, the identities refer to a particular realization of the preference shocks. In the original model, players randomize uniformly on $\left(0,\, \tilde{t}^n\right]$ if and only if $\tilde{t}^n > 0$ (see Proposition 1.1). This implies $e_{ij} \leq \tilde{t}_j^n$ for both types. Then, I impose another constraint on the support of $f_i\left(\varepsilon\right)$:

Assumption 2: $\bar{\varepsilon}_i = t_j^i - \max\left\{e_{nj},\, e_{gj}\right\}$, $i = \{g,\, n\}$ for every match j.

Assumption 2 states that the noise distribution for type g depends on his opponent's characteristics. Moreover, the condition implies $\tilde{t}_j^n > 0$ for any match j and, as a consequence, the non-trivial equilibrium.

Finally, the support of the noise distribution must be well-defined. I introduce Assumption 3 to guarantee $\bar{\varepsilon}_i > \underline{\varepsilon}_i$:

Assumption 3: $\max\left\{e_{nj},\, e_{gj}\right\} < \min\left\{U_{n,j}^W,\, U_{g,j}^W\right\}$ for every match j.

Appendix D. Estimation Results and the Goodness-of-Fit: Relative Effort

I checks if the structural model replicates the key empirical patterns in effort. Fitting this variable in the proposed framework is potentially problematic. If the model is a good approximation of contestants' behavior, the athletes must play mixed strategies. Then, the sum of observed effort levels can differ from its expected value characterized in the model. For this reason, I compare not absolute but relative effort between various groups of players.

Definition 1.10. *Let there be $k \geq 1$ ordered groups, and e_k denotes average expected effort of group k. A chain ratio between groups k and $k-1$ is the ratio of effort levels in these groups:*

$$\rho_{k,\,k-1} = \frac{e_k}{e_{k-1}},\ and\,\rho_{1,\,0} = 1$$

First, I compare actual and predicted chain ratios over different age and ranking groups.[83] The former partition looks as follows:

$$d_k^{age} \in D^{age},\ D^{age} = \{\min_i (age_i)\,,\,20,\,25,\,30,\,\max_i (age_i)\}$$
$$i \in k \Leftrightarrow age_i \in \left[d_k^{age},\, d_{k+1}^{age}\right),\ k = \{1,\,...,\,4\}$$

where k is the group. The partition based on ranking points is given by variable q (see Subsection 4.1). In addition, I trace the total effort in competitions where contestants from two extreme elements of each partition meet.

Further, I divide matches into groups based on contestants' heterogeneity. To approximate this value in the data, I use two approaches:

1. Take absolute differences in contestants' ranking points for every match j:

$$het_j^p = |points_{ij} - points_{-ij}|$$

   In the structural model, I treat players' valuations and skills as functions of their ranking points. As a result, the proposed approach must capture some heterogeneity between players in the data.

2. Take absolute differences in contestants' betting odds for every match j:

[83] For every group, I take the average of expected effort over games and simulations. In the data, I repeat the same procedure for individual effort choices. This determines predicted and empirical chain ratios.

$$het_j^B = |Bet_{ij} - Bet_{-ij}|$$

Betting odds may reflect not only contestants' rankings but also additional information about their relative strength.[84]

Partitions based on contestants' heterogeneity are summarized as follows:

$$d_k^{het,\,l} \in D^{het,\,l},\; l = \{p,\, B\}$$
$$j \in k \Leftrightarrow het_j^l \in \left[d_k^{het,\,l},\, d_{k+1}^{het,\,l}\right),\; k = \left\{1,\, ...,\, \#D^{het,\,l} - 1\right\}$$

where k is the group, p and B refer to the approaches used to approximate heterogeneity in the data.[85]

Figure 1.5.8 contains predicted and actual chain ratios. On average, the model replicates the shapes of the empirical curves well although sometimes the magnitude of predicted peaks and falls is not exactly captured.

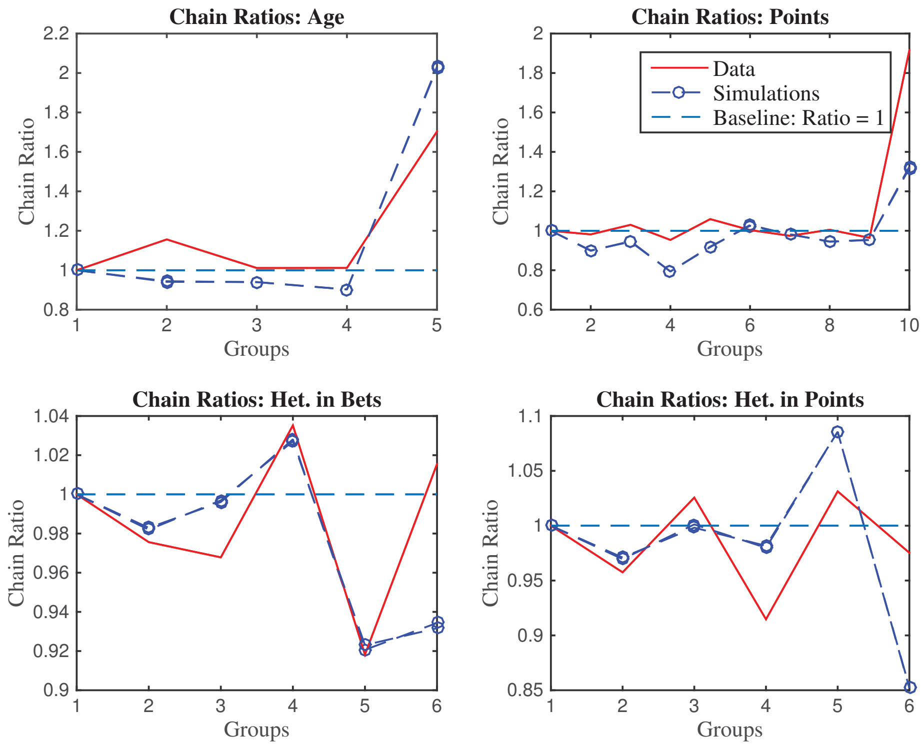
[84] This additional information could be injuries, long recovery, changes in the training schedule etc.

[85] The partitions are based on empirical distributions of het_j^p and het_j^B :

$$D^{het,\,p} = \left\{0,\, 200,\, 500,\, 1000,\, 1600,\, 3000,\, \max_j \left(het_j^p\right)\right\}$$
$$D^{het,\,B} = \left\{0,\, 1,\, 1.93,\, 6,\, 8,\, 11,\, \max_j \left(het_j^B\right)\right\}$$

The thresholds were subject to robustness checks.

Figure 1.5.8: Goodness–of–Fit: Chain Ratios



Note: the points-based partition is driven by variable q (see Subsection 4.1); other partitions look as follows:

$$d_k^{age} \in D^{age},\, D^{age} = \{\min_i (age_i)\,,\, 20,\, 25,\, 30,\, \max_i (age_i)\}$$
$$i \in k \Leftrightarrow age_i \in \left[d_k^{age},\, d_{k+1}^{age}\right)\,,\, k = \{1,\, ...,\, 4\}$$
$$d_k^{het,\, l} \in D^{het,\, l},\, l = \{p,\, B\}$$
$$j \in k \Leftrightarrow het_j^l \in \left[d_k^{het,\, l},\, d_{k+1}^{het,\, l}\right)\,,\, k = \left\{1,\, ...,\, \#D^{het,\, l} - 1\right\}$$
$$D^{het,\, p} = \left\{0,\, 200,\, 500,\, 1000,\, 1600,\, 3000,\, \max_j \left(het_j^p\right)\right\}$$
$$D^{het,\, B} = \left\{0,\, 1,\, 1.93,\, 6,\, 8,\, 11,\, \max_j \left(het_j^B\right)\right\}$$
$$het_j^p = |points_{ij} - points_{-ij}|\,,\, het_j^B = |Bet_{ij} - Bet_{-ij}|$$

where k is the group, p and B refer to the approaches used to approximate heterogeneity in the data; group 5 (10) in subplot “Chain Ratios: Age” (“Chain Ratios: Points”) corresponds to mean expected aggregate effort in the matches where contestants from two extreme elements of the partition, 1 and 4 (1 and 9), meet.

Appendix E. Proofs

Proposition 1.1. *For $t_n \leq 0$ the equilibrium is always trivial. For $t_n > 0$ the equilibrium is unique and non-trivial:*

- *Contestant g randomizes uniformly on $[0,\, t_n]$, and his equilibrium payoff is $\pi_g = t_g - t_n + U_g^L \geq U_g^L$.*
- *Contestant n randomizes uniformly on $(0,\, t_n]$, places the atom of size $p_n^0 = \frac{t_g - t_n}{t_g}$ at $e = 0$, and his equilibrium payoff is $\pi_n = U_n^L \leq \pi_g$.*

Proof. Let $G_i\left(e\right)$ be a probability that contestant i chooses at most e, and $\underline{e}_i$, $\bar{e}_i$, $i = \{g,\, n\}$ are lower and upper bounds of $G_i\left(e\right)$'s support. Denote π_i^W, π_i^L, $i = \{g,\, n\}$ as payoffs player i gets when wins or loses, respectively.

First, I show that non-trivial equilibria cannot be obtained under $t_n \leq 0$. Take contestant n and assume $e_n > 0$ in equilibrium. If n wins, he gets $\pi_n^W\left(e_n\right) = t_n + U_n^L - e_n$. In case of losing, the payoff becomes $\pi_n^L\left(e_n\right) = U_n^L - e_n$, and $\pi_n^W \leq \pi_n^L \;\forall\; e_n > 0$ under $t_n \leq 0$. Since losing is preferred to winning, $e_n > 0$ is dominated by $e_n = 0$, and $e_n > 0$ cannot be an equilibrium. Hence, $G_n\left(0\right) = 1$, and there are no non-trivial equilibria.

Second, I show that a non-trivial equilibrium exists under $t_n > 0$ by constructing it. Consider player n. Let $e_n^1 \neq e_n^2$ be two effort levels from $[\underline{e}_n,\, \bar{e}_n]$. In equilibrium e_n^1, e_n^2 must generate the same expected payoff:

$$G_g(e_n^1)\left[U_n^W - e_n^1\right] + \left(1 - G_g\left(e_w^1\right)\right)\left[U_n^L - e_n^1\right] = G_g(e_n^2)\left[U_n^W - e_n^2\right] + \left(1 - G_g\left(e_n^2\right)\right)\left[U_n^L - e_n^2\right]$$

or rearranging terms and using a definition of t_n:

$$\frac{G_g(e_n^1) - G_g(e_n^2)}{e_n^1 - e_n^2} = \frac{1}{t_n}$$

where the right-hand side is constant and does not depend on players' actions. Then taking $e_n^1 - e_n^2 \to 0$ I get:

$$g_g\left(e\right) = \frac{1}{t_n},\; G_g\left(e\right) = \frac{e}{t_n}$$

Next, I analyze player g and consider $e_g^1 \neq e_g^2$ from $\left[\underline{e}_g,\, \bar{e}_g\right]$. The equilibrium requires:

$$G_n(e_g^1)\left[U_g^W - e_g^1\right] + \left(1 - G_n\left(e_g^1\right)\right)\left[U_g^L - e_g^1\right] = G_n(e_g^2)\left[U_g^W - e_g^2\right] + \left(1 - G_n\left(e_g^2\right)\right)\left[U_g^L - e_g^2\right]$$

and after simplifications:

$$\frac{G_n(e_s^1) - G_n(e_s^2)}{e_s^1 - e_s^2} = \frac{1}{t_s}$$

Again, taking $e_n^1 - e_n^2 \to 0$ delivers $g_n(e) = \frac{1}{t_s}$ and $G_n(e) = \frac{e}{t_s}$. Since both density functions $G_g(e)$ and $G_n(e)$ are continuous, there are no atom points within $\left[\underline{e}_g, \bar{e}_g\right]$ and $[\underline{e}_n, \bar{e}_n]$.

Further I characterize supports of contestants' strategies:

Lemma 1. *$\bar{e}_g = \bar{e}_n = t_n$ in equilibrium.*

Proof. The proof proceeds in three steps:

1. $\bar{e}_i \leq t_n$, $i = \{g, n\}$.

   Take player n and suppose he chooses $e_n = t_n + \varepsilon_n$, $\varepsilon_n > 0$ is small enough. Then it must be:
   $$\pi_n^W(t_n + \varepsilon_n) = U_n^L - \varepsilon_n < \pi_n^W(t_n) = U_n^L$$
   and $e_n = t_n$ dominates $e_n = t_n + \varepsilon_n$. Hence, $e_n > t_n$ is never played in equilibrium.

   Now consider player g and assume $e_g = t_n + \varepsilon_g \in (t_n, t_g)$, $\varepsilon_g > 0$ is small enough. As n never bids above t_n, g wins with certainty and gets $\pi_g^W(t_n) = t_g - t_n - \varepsilon_g + U_g^L > 0$. If g chooses $e_g = t_n$, he also succeeds with probability 1, but bears lower costs:
   $$\pi_g^W(t_n) = t_g - t_n + U_g^L$$
   Hence, $e_g = t_n$ dominates $e_g = t_n + \varepsilon_g$ for any $\varepsilon_g > 0$, and $\bar{e}_g \leq t_n$ follows.

2. $\bar{e}_n = \bar{e}_g$.

   Suppose $\bar{e}_n > \bar{e}_g$ and take player n. The contestant wins with certainty when bids $\tilde{e}_n \in (\bar{e}_g, \bar{e}_n]$. However, $e_n = \bar{e}_g$ also results in player n's success and dominates $e_n \in (\bar{e}_g, \bar{e}_n]$:
   $$\pi_n^W(\tilde{e}_n) = t_n + U_n^L - \tilde{e}_n < \pi_n^W(\bar{e}_g) = t_n + U_n^L - \bar{e}_g \,\forall\, \tilde{e}_n \in (\bar{e}_g, \bar{e}_n]$$
   As a result, $\bar{e}_n > \bar{e}_g$ cannot hold in equilibrium. Similar arguments apply to player g's behavior when $\bar{e}_n < \bar{e}_g$ is assumed.

3. $\bar{e}_n = \bar{e}_g = t_n$.

   Assume $\bar{e}_n = \bar{e}_g = k < t_n$ and consider contestant g. If the player deviates towards $e_g \in (k, t_n]$, he wins with certainty and generates no losses in terms of expected utility. The same is true for contestant n.

As a result, $\bar{e}_n = \bar{e}_g = t_n$ must hold in equilibrium.

□

Lemma 2. *$\underline{e}_g = \underline{e}_n = 0$ in equilibrium.*

Proof. $\underline{e}_i \geq 0$, $i = \{g, n\}$ follows from a feasibility constraint imposed on e_i. Next, I prove two statements:

1. $\underline{e}_g = \underline{e}_n$.

   Assume $\underline{e}_g > \underline{e}_n$ and consider a best response of player g to $e_n \in \left(\underline{e}_n, \underline{e}_g - \varepsilon\right)$, $\varepsilon \in \left(0, \underline{e}_g - \underline{e}_n\right)$. When bidding $e_g = \underline{e}_g$, contestant g wins with certainty and gets $\pi_g^W\left(\underline{e}_g\right) = t_g + U_g^L - \underline{e}_g$. A deviation towards $\hat{e}_g = \underline{e}_g - \eta$, $\eta \in (0, \varepsilon)$ also results in g's success, but leads to higher payoff:

   $$\pi_g^W\left(\hat{e}_g\right) = t_g + U_g^L - \underline{e}_g + \eta > \pi_g^W\left(\underline{e}_g\right) = t_g + U_g^L - \underline{e}_g \,\forall\, \eta \in (0, \varepsilon)$$

   Hence, under $\underline{e}_g > \underline{e}_n$ there exists a profitable deviation contestant g can implement, and this cannot be an equilibrium. Similar arguments apply to player n's behavior when $\underline{e}_g < \underline{e}_n$ is assumed.

2. $\underline{e}_n = 0$. Suppose $\underline{e}_n = l > 0$. If contestant n loses with $e_n = \underline{e}_n$, he gets $\pi_n^L\left(\underline{e}_n\right) = U_n^L - \underline{e}_n < \pi_n^L(0)$. As a result, the player can profitably deviate towards $\underline{e}_n = 0$, and this will constitute an equilibrium.

Two results combined together imply $\underline{e}_g = \underline{e}_n = 0$ in equilibrium.

□

Now I can provide a complete characterization of contestants' strategies:

- $G_g\left(\bar{e}_g\right) = 1$ and $G_g\left(\underline{e}_g\right) = 0$ imply that g's strategy has no atoms, and the player randomizes continuously in $\left[\underline{e}_g, \bar{e}_g\right]$ with a corresponding expected effort $\frac{t_n}{2}$;
- $G_n\left(\bar{e}_n\right) = \frac{t_n}{t_g} < 1$ signals that n's strategy contains an atom at zero, and it must be $G_n\left(\underline{e}_n\right) = \frac{t_g - t_n}{t_g}$. Summing up, player n uses continuous randomization on $(0, t_n]$, places $p_n^0 = \frac{t_g - t_n}{t_g}$ at 0, and his expected effort is $\frac{t_n^2}{2t_g}$;
- Equilibrium payoffs are $\pi_g = t_g - t_n + U_g^L$ and $\pi_n = U_n^L$, and it must be $\pi_g \geq \pi_n$.

Lemma 3. *$t_g \geq t_n$ implies $\pi_g \geq \pi_n$.*

Proof. The inequality $\pi_g \geq \pi_n$ can be rewritten in the following way:

$$\pi_g \geq \pi_n \Leftrightarrow t_g - t_n \geq U_n^L - U_g^L \Leftrightarrow U_g^W - U_n^W \geq 0$$

Several possible outcomes emerge:

1. $U_n^L \leq U_g^L \Rightarrow \pi_g \geq \pi_n$ follows from $t_g \geq t_n$ by definition of contestant types;
2. $U_n^L > U_g^L$:

(a) $U_g^W > U_n^W \Rightarrow t_g \geq t_n$ implies $\pi_g \geq \pi_n$;

(b) $U_g^W < U_n^W \Rightarrow$ it must be:

$$\begin{cases} U_g^W < U_n^W \\ U_g^L < U_n^L \end{cases}$$

Summing these conditions up results in $U_g^W + U_g^L < U_n^W + U_n^L$, or equivalently, $U_g^W - U_n^W < U_n^L - U_g^L$. Combining this constraint with $t_g > t_n$ delivers

$$\begin{cases} U_g^W - U_n^W < U_n^L - U_g^L \\ U_g^W - U_n^W > U_g^L - U_n^L \end{cases}$$

and a set of feasible $\left\{U_g^W - U_n^W\right\}$ is non-empty $\Leftrightarrow U_n^L < U_g^L$, a contradiction.

As a result, $t_g \geq t_n$ implies $\pi_g \geq \pi_n$, and player g never gets less then his competitor n.

$\square$

Since there is no any other candidate equilibrium, uniqueness follows by construction.

$\square$

Proposition 1.2. *For any $\hat{\triangle}_B \in [-1, 1]$ and valuation profile $r \in R$ the designer*

- *Uses both goods and leaves a positive losing prize in dimension A, or*
- *Uses both goods and gives the endowment of A to a winner, or*
- *Does not use dimension B and gives the endowment of A to a winner.*

Proof. Let $r = \{\alpha_g, \alpha_n, \beta_g, \beta_n\}$. A complete characterization of the designer's problem under fixed B^W, B^L looks as follows:

$$\max_{A^W, A^L} \left\{ J\left(A^W, A^L\right) + \eta_W A^W + \eta_L A^L + \lambda\left(1 - A^W - A^L\right)\right\} \tag{1.5.1}$$

Abusing notations, re-define $J\left(A^W, A^L, \hat{B}^W, \hat{B}^L\right) \equiv J\left(\triangle_A, \hat{\triangle}_B\right)$. Let $g = i$, $i = \{1, 2\}$ if and only if $t_i\left(1, \hat{\triangle}_B\right) > t_{-i}\left(1, \hat{\triangle}_B\right)$, which implies

$$\alpha_g + \beta_g \hat{\triangle}_B > \alpha_n + \beta_n \hat{\triangle}_B \Leftrightarrow \alpha_g > \alpha_n + (\beta_n - \beta_g)\hat{\triangle}_B \tag{1.5.2}$$

Assume no good is left out.

Lemma 1. *Problem (1) has no interior solution.*

Proof. Consider a second-order derivative of $J\left(\triangle_A, \hat{\triangle}_B\right)$:

$$\frac{\partial^2 J}{\partial \triangle_A^2}\left(\triangle_A, \hat{\triangle}_B\right) = \frac{\hat{\triangle}_B^2 \left(\beta_n \alpha_g - \beta_g \alpha_n\right)^2}{t_g^2} > 0$$

Then, any $\triangle_A^*$ such that $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A^*, \hat{\triangle}_B\right) = 0$ corresponds to an interior minimum of $J\left(\triangle_A, \hat{\triangle}_B\right)$. $\square$

Corollary 1. $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A, \hat{\triangle}_B\right)$ *strictly increases in* $\triangle_A$.

Let $\tilde{\triangle}_A$ be a prize spread in dimension A such that contestants' types, t_n and t_g, equalize given $\hat{\triangle}_B$:

$$t_g\left(\triangle_A, \hat{\triangle}_B\right) = t_n\left(\triangle_A, \hat{\triangle}_B\right) \Leftrightarrow \tilde{\triangle}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n}\hat{\triangle}_B$$

One can show that $\tilde{\triangle}_A$ is never feasible for $\alpha_g < \alpha_n$:

$$\tilde{\triangle}_A \leq 1 \Leftrightarrow \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n}\hat{\triangle}_B \leq 1 \Leftrightarrow \alpha_g \leq \alpha_n + (\beta_n - \beta_g)\hat{\triangle}_B$$

where the latter inequality contradicts condition 1.5.2. Hence, a necessary condition for $\triangle_A = \tilde{\triangle}_A$ being feasible is $\alpha_g > \alpha_n$. Further, we will work only with this restriction.

Lemma 2. $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A, \hat{\triangle}_B\right)$ *has a jump point at* $\triangle_A = \tilde{\triangle}_A$ *for any* $\alpha_g \neq \alpha_n$.

Proof. To prove the statement, compute the limits of $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right)$ when it approaches $\triangle_A = \tilde{\triangle}_A$ from the left and from the right:

$$\begin{aligned}\lim_{\triangle_A \to - \tilde{\triangle}_A} \tfrac{\partial J}{\partial \triangle_A}\left(\cdot\right) &= \tfrac{3\alpha_g - \alpha_n}{2}\\ \lim_{\triangle_A \to + \tilde{\triangle}_A} \tfrac{\partial J}{\partial \triangle_A}\left(\cdot\right) &= \tfrac{3\alpha_n - \alpha_g}{2}\end{aligned}$$

The limits are equal if and only if $\alpha_g = \alpha_n$. Thus, $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right)$ has a jump point at $\triangle_A = \tilde{\triangle}_A$ for any $\alpha_g \neq \alpha_n$.

□

Assume $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right)$ is a right-continuous function.[86] Here and after, $\frac{\partial J}{\partial \triangle_A}^{-}\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right)$ and $\frac{\partial J}{\partial \triangle_A}^{+}\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right)$ denote derivatives of $J\left(\cdot\right)$ to the left and to the right of $\triangle_A = \tilde{\triangle}_A$, respectively. Given $\hat{\triangle}_B$, contestants' types are well-defined if and only if

$$t_j\left(\triangle_A,\, \hat{\triangle}_B\right) \geq 0 \Leftrightarrow \triangle_A^j \geq -\frac{\hat{\triangle}_B \beta_j}{\alpha_j},\ j = \{g,\, n\}$$

Depending on players' preferences, $\triangle_A^j$ can be to the left or to the right of $\tilde{\triangle}_A$.

Definition. $\tilde{\triangle}_A$ *is type-feasible if and only if* $\max\left\{\triangle_A^g,\, \triangle_A^n\right\} < \tilde{\triangle}_A$.[87]

Lemma 3. *For any valuation profile such that $\tilde{\triangle}_A$ is not type-feasible it is optimal to give the endowment of good A to a winner ($\triangle_A = 1$).*

Proof. First, we show that no type-feasibility holds if and only if $\triangle_A^n \geq \tilde{\triangle}_A$. Expanding $\triangle_A^n \geq \tilde{\triangle}_A$ delivers:

$$\triangle_A^n \geq \tilde{\triangle}_A \Leftrightarrow \begin{cases}\frac{\beta_g}{\alpha_g} > \frac{\beta_n}{\alpha_n}\, if\, \hat{\triangle}_B \geq 0\\ \frac{\beta_g}{\alpha_g} < \frac{\beta_n}{\alpha_n}\, if\, \hat{\triangle}_B < 0\end{cases}$$

where conditions imposed on contestants' marginal rates of substitution correspond to $\max\left\{\triangle_A^g,\, \triangle_A^n\right\} = \triangle_A^n$ for given $\hat{\triangle}_B$. When $\triangle_A = \triangle_A^n$, the designer ends up with zero expected aggregate effort, and this allocation is strictly dominated by $\triangle_A > \triangle_A^n$. Hence, under $\triangle_A^n \geq \tilde{\triangle}_A$, the choice set is bounded by $\triangle_A \in (\triangle_A^n,\, 1]$.

Second, consider the derivative $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right)$ at $\triangle_A = \triangle_A^n$:

$$\frac{\partial J}{\partial \triangle_A}\left(\triangle_A^n,\, \hat{\triangle}_B\right) = \frac{\alpha_n}{2} > 0$$

[86] Continuity of $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right)$ at any $\triangle_A \neq \tilde{\triangle}_A$ can be proved easily.

[87] Type-feasibility is introduced to avoid confusions with a standard feasibility notion, which means $\triangle_A \in [0,\, 1]$.

Since $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right)$ is positive at $\triangle_A = \triangle_A^n$ and $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right)$ strictly increases in $\triangle_A$ (***Corollary 1***), it must be $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right) > 0$ for any $\triangle_A \in (\triangle_A^n,\, 1]$. As a result, $J(\cdot)$ strictly increases in $\triangle_A$, and $\triangle_A = 1$ is optimal.

□

Lemma 4. *For any valuation profile such that $\tilde{\triangle}_A$ is type-feasible the objective function $J(\cdot)$ strictly increases in $\triangle_A$ for $\triangle_A \in \left[\triangle_A^g,\, \tilde{\triangle}_A\right)$. $J(\cdot)$ strictly decreases or is non-monotone in $\triangle_A$ for $\triangle_A \in \left[\tilde{\triangle}_A,\, 1\right]$ if and only if $\alpha_g > 3\alpha_n$; otherwise, $J(\cdot)$ strictly increases in $\triangle_A$ for $\triangle_A \in \left[\tilde{\triangle}_A,\, 1\right]$.*

Proof. In the beginning, we prove that type-feasibility holds if and only if $\triangle_A^g < \tilde{\triangle}_A$:

$$\triangle_A^g < \tilde{\triangle}_A \Leftrightarrow \begin{cases} \frac{\beta_g}{\alpha_g} < \frac{\beta_n}{\alpha_n}\, if\, \hat{\triangle}_B \geq 0 \\ \frac{\beta_g}{\alpha_g} > \frac{\beta_n}{\alpha_n}\, if\, \hat{\triangle}_B < 0 \end{cases}$$

where latter inequalities guarantee $\max\{\triangle_A^g,\, \triangle_A^n\} = \triangle_A^g$ for any $\hat{\triangle}_B$. Allocating $\triangle_A = \triangle_A^g$ results in zero expected aggregate effort, and the designer's choice set is bounded by $\triangle_A \in (\triangle_A^g,\, 1]$.

Since $\tilde{\triangle}_A$ is type-feasible and $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right)$ has a jump point at $\triangle_A = \hat{\triangle}_A$, one must analyze the intervals $\triangle_A \in \left[\triangle_A^g,\, \tilde{\triangle}_A\right)$ and $\triangle_A \in \left[\tilde{\triangle}_A,\, 1\right]$ separately. Consider $\triangle_A \in \left[\triangle_A^g,\, \tilde{\triangle}_A\right)$ and evaluate $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A,\, \hat{\triangle}_B\right)$ at $\triangle_A = \triangle_A^g$:

$$\frac{\partial J}{\partial \triangle_A}\left(\triangle_A^g,\, \hat{\triangle}_B\right) = \frac{\alpha_g}{2} > 0$$

Given $\frac{\partial J}{\partial \triangle_A}\left(\triangle_A^g,\, \hat{\triangle}_B\right) > 0$ and ***Corollary 1***, $J(\cdot)$ must strictly increase in $\triangle_A$ for any $\triangle_A \in \left[\triangle_A^g,\, \tilde{\triangle}_A\right)$.

Now, take $\triangle_A \in \left[\tilde{\triangle}_A,\, 1\right]$ and recall ***Lemma 2***:

$$\frac{\partial J}{\partial \triangle_A}^+\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right) = \frac{3\alpha_n - \alpha_g}{2}$$

and $\frac{\partial J}{\partial \triangle_A}^+(\cdot) \geq 0$ if and only if $\alpha_g \leq 3\alpha_n$. Given ***Corollary 1***, under $\alpha_g \leq 3\alpha_n$ and $\frac{\partial J}{\partial \triangle_A}\left(1,\, \hat{\triangle}_B\right) > 0$ there exists a unique value $\bar{\triangle}_A \in \left(\tilde{\triangle}_A,\, 1\right)$ such that $\frac{\partial J}{\partial \triangle_A}\left(\bar{\triangle}_A,\, \hat{\triangle}_B\right) = 0$ and $J(\cdot)$ strictly decreases (increases) in $\triangle_A$ for $\triangle_A \in \left(\tilde{\triangle}_A,\, \bar{\triangle}_A\right)$ $\left(\triangle_A \in \left(\bar{\triangle}_A,\, 1\right]\right)$. When $\alpha_g \leq 3\alpha_n$ and $\frac{\partial J}{\partial \triangle_A}\left(1,\, \hat{\triangle}_B\right) < 0$, the objective function $J(\cdot)$ strictly decreases in $\triangle_A$ for $\triangle_A \in \left(\tilde{\triangle}_A,\, 1\right]$.

□

To sum up, type-feasibility requires:

$$\begin{cases} \alpha_g > \max\left\{\alpha_n + (\beta_n - \beta_g)\hat{\triangle}_B,\, \alpha_n,\, \alpha_n\frac{\beta_g}{\beta_n}\right\} \; if\, \hat{\triangle}_B \geq 0 \\ \begin{cases} \alpha_g \in \left(\alpha_n + (\beta_n - \beta_g)\hat{\triangle}_B,\, \alpha_n\frac{\beta_g}{\beta_n}\right) \\ \beta_g > \beta_n \end{cases} \quad if\, \hat{\triangle}_B < 0 \end{cases} \tag{1.5.3}$$

Condition $\alpha_g > 3\alpha_n$ is necessary (but not sufficient) to make $\triangle_A < 1$ optimal. When it holds and $\tilde{\triangle}_A$ is type-feasible, only two candidate prize allocations with both goods used are left and must be compared directly (***Lemma 1*** excludes interior solutions):

$$\left(1,\, \hat{\triangle}_B\right) \; or \; \left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right)$$

The designer cannot change the prize allocation in dimension B and may prefer to leave this item out.

Lemma 5. *If dimension B is left out, it is optimal to give the endowment of good A to a winner ($\triangle_A = 1$).*

Proof. When there is no dimension B, $\frac{\partial J}{\partial \triangle_A}(\triangle_A,\, 0)$ looks as follows:

$$\frac{\partial J}{\partial \triangle_A}(\triangle_A,\, 0) = \frac{\alpha_n(\alpha_n + \alpha_g)}{2\alpha_g} I_{\{\alpha_g > \alpha_n\}} + \frac{\alpha_g(\alpha_g + \alpha_n)}{2\alpha_n}\left[1 - I_{\{\alpha_g > \alpha_n\}}\right] > 0$$

where $I_{\{\alpha_g > \alpha_n\}} = 1$ if $\alpha_g > \alpha_n$ and $I_{\{\alpha_g > \alpha_n\}} = 0$, otherwise. Given that $J(\cdot)$ strictly increases in $\triangle_A$, it is optimal to choose $\triangle_A = 1$ and use good A's endowment completely.

□

Here and after, a prize schedule where dimension B is omitted is denoted as $(1,\, 0)_{\{\alpha_g > \alpha_n\}}$.

Now, we analyze the cases when type-feasibility holds and the objective function $J(\cdot)$ strictly decreases or is non-monotone in $\triangle_A$ for $\triangle_A \in \left(\tilde{\triangle}_A,\, 1\right]$, i.e. $\alpha_g > 3\alpha_n$:

1. $\tilde{\triangle}_A \leq -1$, i.e. $\tilde{\triangle}_A$ is not feasible.

   This condition holds if and only if

$$\alpha_g \leq \alpha_n - (\beta_n - \beta_g)\hat{\triangle}_B \tag{1.5.4}$$

   For $\hat{\triangle}_B \geq 0$, (1.5.4) always has a non-empty intersection with (1.5.3) if and only if

$$\begin{cases} \alpha_g \in \left(3\alpha_n,\, \alpha_n - (\beta_n - \beta_g)\hat{\triangle}_B\right) \\ \alpha_n \in \left[0,\, \min\left\{\frac{(\beta_g - \beta_n)}{2}\hat{\triangle}_B,\, \beta_n\hat{\triangle}_B\right\}\right) \\ \beta_g > \beta_n > 0 \end{cases} \tag{1.5.5}$$

   When $\hat{\triangle}_B < 0$, (1.5.4) combined with (1.5.3) always delivers an empty set for $\beta_g > \beta_n$. Hence, under $\tilde{\triangle}_A \leq -1$, we study only the case of $\hat{\triangle}_B \geq 0$.

Next, one must compare $J\left(-1,\, \hat{\triangle}_B\right)$, $J\left(1,\, \hat{\triangle}_B\right)$, and $J\left(1,\, 0\right)_{\{\alpha_g>\alpha_n\}}$. In the beginning, consider $J\left(-1,\, \hat{\triangle}_B\right)$ and $J\left(1,\, \hat{\triangle}_B\right)$:

$$J\left(-1,\, \hat{\triangle}_B\right) > J\left(1,\, \hat{\triangle}_B\right) \Leftrightarrow \alpha_n\alpha_g^2 + \left(\alpha_n^2 + \beta_n^2\hat{\triangle}_B^2\right)\alpha_g - \hat{\triangle}_B^2\alpha_n\beta_g\left(\beta_g + 2\beta_n\right) > 0 \quad (1.5.6)$$

The corresponding square equation solved for α_g always has two real roots, and condition (1.5.6) holds if and only if

$$\alpha_g > r_1 > 0 \quad (1.5.7)$$

where

$$r_1 = \frac{-\left(\alpha_n^2 + \beta_n^2\hat{\triangle}_B^2\right) + \sqrt{\left(\alpha_n^2 + \beta_n^2\hat{\triangle}_B^2\right)^2 + 4\hat{\triangle}_B^2\alpha_n^2\beta_g\left(\beta_g + 2\beta_n\right)}}{2\alpha_n}$$

Conditions (1.5.5) and (1.5.7) define a non-empty set of α_g's if and only if

$$\alpha_n \in [0,\, q_1) \quad (1.5.8)$$

where

$$q_1 = \frac{\left(\beta_n - 3\beta_g\right) + \sqrt{\left(\beta_n + \beta_g\right)\left(9\beta_g - 7\beta_n\right)}}{4}\hat{\triangle}_B < \min\left\{\frac{\left(\beta_g - \beta_n\right)}{2}\hat{\triangle}_B,\, \beta_n\hat{\triangle}_B\right\}$$

Further, compare $J\left(-1,\, \hat{\triangle}_B\right)$ and $J\left(1,\, 0\right)_{\{\alpha_g>\alpha_n\}}$:

$$J\left(-1,\, \hat{\triangle}_B\right) > J\left(1,\, 0\right)_{\{\alpha_g>\alpha_n\}} \Leftrightarrow$$

$$\left(2\alpha_n - \beta_n\hat{\triangle}_B\right)\alpha_g^2 + \left(2\alpha_n^2 - \left(2\alpha_n - \beta_n\hat{\triangle}_B\right)\left(\beta_n + \beta_g\right)\hat{\triangle}_B\right)\alpha_g - \alpha_n^2\beta_g\hat{\triangle}_B > 0 \quad (1.5.9)$$

where $\left(2\alpha_n - \beta_n\hat{\triangle}_B\right) < 0$ for any $\alpha_n \in [0,\, q_1)$. The underlying square equation always has two real roots, $\tilde{r}_1$ and $\tilde{r}_2$, and $0 < \tilde{r}_1 \leq \tilde{r}_2$ when $\alpha_n \in [0,\, q_1)$. Then, the designer prefers $\left(-1,\, \hat{\triangle}_B\right)$ to $\left(1,\, 0\right)_{\{\alpha_g>\alpha_n\}}$ if and only if $\alpha_g \in (\tilde{r}_1,\, \tilde{r}_2)$, and this set combined with (1.5.5) and (1.5.7) must deliver a non-empty intersection.

Lemma 6. *For any preference structure such that $\tilde{\triangle}_A = -1$ is feasible and $J\left(-1,\, \hat{\triangle}_B\right) > J\left(1,\, \hat{\triangle}_B\right)$, the designer never ignores good B.*

Proof. To prove the lemma, $\tilde{r}_1 < 3\alpha_n$ and $\tilde{r}_2 > \alpha_n - \hat{\triangle}_B\left(\beta_n - \beta_g\right)$ for any $\alpha_n \in [0,\, q_1)$ are sufficient. Consider the $\tilde{r}_1 < 3\alpha_n$ condition. Solving the underlying equation $\tilde{r}_1 - 3\alpha_n = 0$ with respect to α_n delivers three roots:

$$k_1 = \frac{\beta_n \hat{\triangle}_B}{2} > q_1$$
$$k_{2,3} = \frac{\left(7\beta_g + 15\beta_n \mp \sqrt{49\beta_g^2 - 78\beta_g\beta_n - 63\beta_n^2}\right)}{48}\hat{\triangle}_B$$

If $\beta_g \in \left(\beta_n,\ \frac{\left(39+48\sqrt{2}\right)}{49}\beta_n\right)$, roots k_2 and k_3 are complex. Then, $\tilde{r}_1 - 3\alpha_n < 0$ for $\alpha_n \in [0,\ k_1)$ and $\tilde{r}_1 - 3\alpha_n \geq 0$ for $\alpha_n \geq k_1$. Imposing constraint (1.5.8), we get $\tilde{r}_1 - 3\alpha_n < 0$ for any $\alpha_n \in [0,\ q_1)$.

When $\beta_g \geq \frac{\left(39+48\sqrt{2}\right)}{49}\beta_n$, both k_2 and k_3 become real and positive. Moreover:

$$k_3 > k_1 > q_1\ for\ any\ \beta_g \geq \frac{\left(39+48\sqrt{2}\right)}{49}\beta_n$$
$$k_2 > k_1 > q_1\ for\ \beta_g \in \left[\frac{\left(39+48\sqrt{2}\right)}{49}\beta_n,\ 3\beta_n\right)\ and\ k_2 \leq k_1\ for\ \beta_g \geq 3\beta_n$$

With $\beta_g \in \left[\frac{\left(39+48\sqrt{2}\right)}{49}\beta_n,\ 3\beta_n\right)$, the inequality $\tilde{r}_1 - 3\alpha_n < 0$ holds for $\alpha_n \in [0,\ k_1)$. Combining this with constraint (1.5.8) results in $\tilde{r}_1 - 3\alpha_n < 0$ for any $\alpha_n \in [0,\ q_1)$. Under $\beta_g \geq 3\beta_n$, $\tilde{r}_1 - 3\alpha_n < 0$ ($\tilde{r}_1 - 3\alpha_n \geq 0$) holds for $\alpha_n \in [0,\ k_2)$ ($\alpha_n \in [k_2,\ k_1)$). We analyze q_1 and k_2 as functions of β_g. First, compare $q_1\left(3\beta_g\right)$ and $k_2\left(3\beta_g\right)$:

$$q_1\left(3\beta_g\right) = \left(\sqrt{5}-2\right)\beta_n\hat{\triangle}_B < \frac{\beta_n\hat{\triangle}_B}{2} = k_2\left(3\beta_g\right)$$

Second, compute derivatives of q_1 and k_2 with respect to β_g:

$$\frac{\partial q_1}{\partial \beta_g} = \frac{9\beta_g + \beta_n - 3\sqrt{(\beta_n+\beta_g)(9\beta_g - 7\beta_n)}}{4\sqrt{(\beta_n+\beta_g)(9\beta_g-7\beta_n)}}\hat{\triangle}_B > 0$$
$$\frac{\partial k_2}{\partial \beta_g} = \frac{\left(-49\beta_g + 39\beta_n + 7\sqrt{49\beta_g^2 - 78\beta_g\beta_n - 63\beta_n^2}\right)}{48\sqrt{49\beta_g^2 - 78\beta_g\beta_n - 63\beta_n^2}}\hat{\triangle}_B < 0$$

Hence, q_1 (k_2) strictly increases (decreases) in β_g for any $\beta_g,\ \beta_n,\ \hat{\triangle}_B \geq 0$, and the equation $k_2 - q_1 = 0$ must have at most one root in $\beta_g \geq 3\beta_n$. Finally, take a limit of $(k_2 - q_1)$ for $\beta_g \to \infty$:

$$\lim_{\beta_g \to \infty}(k_2 - q_1) = \frac{2\beta_n\hat{\triangle}_B}{21} > 0$$

Given strict monotonicity of k_1 and q_1, $k_2\left(3\beta_g\right) > q_1\left(3\beta_g\right)$, and $\lim_{\beta_g\to\infty}(k_1 - q_1) > 0$, there does not exist $\tilde{\beta}_g > 3\beta_n$ such that $\left(k_1\left(\tilde{\beta}_g\right) - q_1\left(\tilde{\beta}_g\right) = 0\right)$. Thus, $k_2 > q_1$ for any $\beta_g \geq 3\beta_n$. Since $\tilde{r}_1 - 3\alpha_n < 0$ for any $\alpha_n \in [0,\ k_2)$, imposing constraint (1.5.8) delivers the first statement of the lemma.

Next, consider the second inequality, $\tilde{r}_2 > \alpha_n - \hat{\triangle}_B(\beta_n - \beta_g)$. Solving $\tilde{r}_2 = \alpha_n - \hat{\triangle}_B(\beta_n - \beta_g)$ with respect to α_n delivers four roots:

$$d_{1,2} = \tfrac{\hat{\triangle}_B}{8}\left(5\beta_n - \beta_g \mp \sqrt{(\beta_g + \beta_n)(9\beta_g - 7\beta_n)}\right)$$
$$d_3 = \tfrac{\beta_n \hat{\triangle}_B}{2},\ d_4 = \beta_n \hat{\triangle}_B$$

where $d_1 < 0 < d_2 < d_3 < d_4$ and $q_1 \in (0,\, d_2)$ for any $\beta_g \geq 0$. The inequality $\tilde{r}_2 > \alpha_n - \hat{\triangle}_B(\beta_n - \beta_g)$ holds for $\alpha_n \in [0,\, d_2)$. Combining this with condition (1.5.8), $\tilde{r}_2 > \alpha_n - \hat{\triangle}_B(\beta_n - \beta_g)$ for any $\alpha_n \in [0,\, q_1)$ comes out.

Thus, feasibility of $\tilde{\triangle}_A = -1$ and $J\left(-1,\, \hat{\triangle}_B\right) > J\left(1,\, \hat{\triangle}_B\right)$ imply $J\left(-1,\, \hat{\triangle}_B\right) > J(1,\, 0)_{\{\alpha_g > \alpha_n\}}$.

$\square$

To sum up, $\triangle_A = -1$ is optimal if and only if

$$\begin{cases} \alpha_g \in \left(\max\{r_1,\, 3\alpha_n\},\, \alpha_n - (\beta_n - \beta_g)\hat{\triangle}_B\right) \\ \alpha_n \in [0,\, q_1) \\ \beta_g > \beta_n > 0 \end{cases}$$

When $\alpha_n \in \left(q_1,\, \min\left\{\frac{(\beta_g - \beta_n)}{2}\hat{\triangle}_B,\, \beta_n\hat{\triangle}_B\right\}\right)$, the set $\left(r_1,\, \alpha_n - (\beta_n - \beta_g)\hat{\triangle}_B\right)$ becomes empty. In words, the $\left(1,\, \hat{\triangle}_B\right)$ bundle strictly dominates the $\left(-1,\, \hat{\triangle}_B\right)$ allocation. To complete the analysis, one must compare $J\left(1,\, \hat{\triangle}_B\right)$ and $J(1,\, 0)_{\{\alpha_g > \alpha_n\}}$ directly:

$$J\left(1,\, \hat{\triangle}_B\right) > J(1,\, 0)_{\{\alpha_g > \alpha_n\}} \Leftrightarrow \beta_n \alpha_g^2 + \beta_n\left(2\alpha_n + \hat{\triangle}_B(\beta_n + \beta_g)\right)\alpha_g - \alpha_n^2\beta_g > 0 \quad (1.5.10)$$

The underlying square equation always has two real roots, $\hat{r}_1$ and $\hat{r}_2$, $\hat{r}_2 < 0 < \hat{r}_1$. Condition (1.5.10) holds if and only if $\alpha_g > \hat{r}_1$.

Lemma 7. *For any preference structure such that $\tilde{\triangle}_A \leq -1$, type-feasibility holds and $J\left(-1,\, \hat{\triangle}_B\right) < J\left(1,\, \hat{\triangle}_B\right)$, the designer never leaves good B out.*

Proof. To prove the lemma, it is sufficient to show that $\hat{r}_1 < \alpha_n$ for any $\alpha_n \in \left[0,\, \beta_n\hat{\triangle}_B\right)$. The corresponding equation $\hat{r}_1 - \alpha_n = 0$ solved for α_n has two roots:

$$c_1 = 0,\ c_2 = \frac{\hat{\triangle}_B \beta_n(\beta_n + \beta_g)}{\beta_g - 3\beta_n}$$

When $\beta_g < 3\beta_n$, the root c_2 is negative, and $\hat{r}_1 - \alpha_n < 0$ holds for any $\alpha_n \geq 0$. If $\beta_g > 3\beta_n$, c_2 becomes positive, and $c_2 > \hat{\triangle}_B\beta_n$. Then, the inequality $\hat{r}_1 - \alpha_n < 0$ is satisfied for any $\alpha_n \in [0,\, c_2)$. Imposing $\alpha_n \in \left[0,\, \beta_n\hat{\triangle}_B\right)$, we get $\hat{r}_1 < \alpha_n$ for any $\alpha_n \in \left[0,\, \beta_n\hat{\triangle}_B\right)$. Thus, $\tilde{\triangle}_A \leq -1$, type-feasibility and $J\left(-1,\, \hat{\triangle}_B\right) < J\left(1,\, \hat{\triangle}_B\right)$ imply $J\left(1,\, \hat{\triangle}_B\right) > J\left(1,\, 0\right)_{\{\alpha_g > \alpha_n\}}$.

□

2. $\tilde{\triangle}_A \in (-1,\, 1)$, i.e. $\tilde{\triangle}_A$ is feasible.

   This is the case if and only if

$$\alpha_g > \alpha_n - (\beta_n - \beta_g)\,\hat{\triangle}_B \tag{1.5.11}$$

Condition (1.5.11) has a non-empty intersection with (1.5.3) and $\alpha_g > 3\alpha_n$ if and only if

$$\begin{cases} \alpha_g > \max\left\{\alpha_n + \hat{\triangle}_B \max\left\{\beta_n - \beta_g,\, \beta_g - \beta_n\right\},\, 3\alpha_n,\, \alpha_n\frac{\beta_g}{\beta_n}\right\} & if\ \hat{\triangle}_B \geq 0 \\ \begin{cases} \alpha_g \in \left(\max\left\{\alpha_n + (\beta_n - \beta_g)\,\hat{\triangle}_B,\, 3\alpha_n\right\},\, \alpha_n\frac{\beta_g}{\beta_n}\right) \\ \alpha_n > -\beta_n\hat{\triangle}_B \\ \beta_g > 3\beta_n \end{cases} & if\ \hat{\triangle}_B < 0 \end{cases} \tag{1.5.12}$$

In the beginning, consider the case of $\hat{\triangle}_B \geq 0$. The designer prefers a bundle with $\triangle_A = \tilde{\triangle}_A \in (-1,\, 1)$ to the $\left(1,\, \hat{\triangle}_B\right)$ allocation if and only if

$$J\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right) > J\left(1,\, \hat{\triangle}_B\right) \Leftrightarrow \left(\beta_n\hat{\triangle}_B - \alpha_n\right)\alpha_g^2 + \hat{\triangle}_B l_1 \alpha_g + \alpha_n l_2 > 0 \tag{1.5.13}$$

where

$$\begin{gathered} l_1 = \beta_n\hat{\triangle}_B\,(\beta_g - \beta_n) - \alpha_n\,(\beta_n + 3\beta_g) \\ l_2 = \alpha_n\left(\alpha_n + \beta_g\hat{\triangle}_B\right) + 2\hat{\triangle}_B\left(\alpha_n\beta_n - \beta_g^2\right) + \hat{\triangle}_B^2\beta_n\,(\beta_n + \beta_g) \end{gathered}$$

Lemma 8. *For $\alpha_n \in \left(0,\, \beta_n\hat{\triangle}_B\right)$ and $\hat{\triangle}_B \geq 0$ there exists α_g^* such that $J\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right) > J\left(1,\, \hat{\triangle}_B\right)$ for any $\alpha_g > \alpha_g^*$.*

Proof. The equation corresponding to condition (1.5.13) always has two real roots:

$$s_1 = \alpha_n + (\beta_n - \beta_g)\,\hat{\triangle}_B,\; s_2 = \frac{\alpha_n\left(\alpha_n + 2\beta_g\hat{\triangle}_B + \beta_n\hat{\triangle}_B\right)}{\beta_n\hat{\triangle}_B - \alpha_n}$$

where s_1 coincides with the lower bound of set (1.5.2). First, take $\alpha_n \in \left(0,\, \beta_n \hat{\triangle}_B\right)$ when both s_1 and s_2 are positive. If $\beta_g \geq \beta_n$, it is always the case that $s_2 > s_1$, and inequality (1.5.13) holds if and only if

$$\alpha_g \in [0,\, s_1) \cup (s_2,\, \infty)$$

Combining this with condition (1.5.12) for $\hat{\triangle}_B \geq 0$, define $\alpha_g^* = \max\left\{\alpha_n + \hat{\triangle}_B \left(\beta_g - \beta_n\right),\, 3\alpha_n,\, \alpha_n \frac{\beta_g}{\beta_n},\, s_2\right\}$. Given $\alpha_n \in \left(0,\, \beta_n \hat{\triangle}_B\right)$ and inequality (1.5.13), it must be $J\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right) > J\left(1,\, \hat{\triangle}_B\right)$ for any $\alpha_g > \alpha_g^*$.

Next, consider $\beta_g < \beta_n$. If $\alpha_n \in \left(0,\, \frac{\beta_n - \beta_g}{2} \hat{\triangle}_B\right]$, it must be $s_2 < s_1$, and condition (1.5.13) implies $J\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right) > J\left(1,\, \hat{\triangle}_B\right)$. When $\alpha_n \in \left(\frac{\beta_n - \beta_g}{2} \hat{\triangle}_B,\, \beta_n \hat{\triangle}_B\right)$, the outcome is exactly the same as in case of $\beta_g \geq \beta_n$. Thus, choosing $\alpha_g^* = \max\left\{\alpha_n + \hat{\triangle}_B \left(\beta_g - \beta_n\right),\, 3\alpha_n,\, \alpha_n \frac{\beta_g}{\beta_n},\, s_2,\, s_1\right\}$ for $\alpha_n \in \left(0,\, \beta_n \hat{\triangle}_B\right)$, we get the statement of the lemma.

Finally, take $\alpha_n > \beta_n \hat{\triangle}_B$, which results in $s_2 < 0$. Then, condition (1.5.13) holds if and only if $\alpha_g \in [0,\, s_1)$, and this contradicts (1.5.12). Hence, the $\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right)$ bundle is never preferred to the $\left(1,\, \hat{\triangle}_B\right)$ allocation when $\alpha_n > \beta_n \hat{\triangle}_B$ and $\hat{\triangle}_B \geq 0$.

□

Lemma 9. *For any preference structure such that* $J\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right) > J\left(1,\, \hat{\triangle}_B\right)$ *and* $\hat{\triangle}_B \geq 0$ *the designer never leaves good* B *out.*

Proof. The designer prefers the $\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right)$ bundle to the $(1,\, 0)_{\{\alpha_g > \alpha_n\}}$ allocation if and only if

$$J\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right) > J\left(1,\, 0\right)_{\{\alpha_g > \alpha_n\}} \Leftrightarrow \left(2\beta_n \hat{\triangle}_B - \alpha_n\right) \alpha_g^2 - \beta_g \alpha_n \hat{\triangle}_B \alpha_g + \alpha_n^3 > 0 \tag{1.5.14}$$

where $\mu_1 = 2\beta_n \hat{\triangle}_B - \alpha_n > 0$ for $\alpha_n \in \left(0,\, \beta_n \hat{\triangle}_B\right)$ and $\hat{\triangle}_B \geq 0$. Two cases emerge:

- $\beta_g \geq \beta_n \Rightarrow$ inequality (1.5.14) has two real roots, $\bar{r}_1$ and $\bar{r}_2$, where $\bar{r}_1 < \bar{r}_2$ and $\bar{r}_1 > 0$ for $\alpha_n \in \left(0,\, \beta_n \hat{\triangle}_B\right)$. Then, the designer prefers $\left(\tilde{\triangle}_A,\, \hat{\triangle}_B\right)$ to $(1,\, 0)_{\{\alpha_g > \alpha_n\}}$ if and only if $\alpha_g \in [0,\, \bar{r}_1) \cup (\bar{r}_2,\, \infty)$. Now, consider how $\bar{r}_2$ relates to s_2 defined in ***Lemma 8.*** Solving $(\bar{r}_2 - s_2 = 0)$ with respect to α_n delivers two roots:
$$\bar{r}_2 - s_2 = 0 \Leftrightarrow \alpha_n^{1,\,2} = \hat{\triangle}_B \left(\beta_n \pm \sqrt{2\beta_n \left(\beta_g + \beta_n\right)}\right)$$

where $\alpha_n^1 > 0 > \alpha_n^2$ for $\beta_g \geq \beta_n$ and $\alpha_n^1 > \beta_n \hat{\triangle}_B$. The inequality $\bar{r}_2 < s_2$ holds for any $\alpha_n \in [0, \alpha_n^1)$; given $\alpha_n^1 > \beta_n \hat{\triangle}_B$, it must be $\bar{r}_2 < s_2$ under $\alpha_n \in \left(0, \beta_n \hat{\triangle}_B\right)$. Hence, $\alpha_g > s_2$ implies $\alpha_g > \bar{r}_2$, i.e. for any preference structure such that $\beta_g \geq \beta_n$ and $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J\left(1, \hat{\triangle}_B\right)$ hold the designer prefers to use both goods.

- $\beta_g < \beta_n \Rightarrow$ roots of condition (1.5.14) are real if and only if

$$\alpha_n \in \left(0, \hat{\triangle}_B \left(\beta_n - \sqrt{\beta_n^2 - \beta_g^2}\right)\right) \cup \left(\hat{\triangle}_B \left(\beta_n + \sqrt{\beta_n^2 - \beta_g^2}\right), \infty\right) \tag{1.5.15}$$

Otherwise, $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J(1, 0)_{\{\alpha_g > \alpha_n\}}$ holds for any α_g. Suppose (1.5.15) is satisfied. Consider how $\bar{r}_2$ and $\left(\alpha_n + \hat{\triangle}_B(\beta_n - \beta_g)\right)$, a lower bound of α_g from condition (1.5.12), locate relative to each other. The equation $\left\{\bar{r}_2 = \alpha_n + \hat{\triangle}_B(\beta_n - \beta_g)\right\}$ has two roots: $\alpha_n = \frac{2\beta_n \hat{\triangle}_B(\beta_n - \beta_g)}{\beta_g - 3\beta_n} < 0$ and $\alpha_n = 2\beta_n \hat{\triangle}_B > \beta_n \hat{\triangle}_B$ – and $\bar{r}_2 < \alpha_n + \hat{\triangle}_B(\beta_n - \beta_g)$ for any $\alpha_n \in \left[0, 2\beta_n \hat{\triangle}_B\right)$. Hence, $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J\left(1, \hat{\triangle}_B\right)$ implies $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J(1, 0)_{\{\alpha_g > \alpha_n\}}$ for $\alpha_n \in \left(0, \beta_n \hat{\triangle}_B\right)$, $\beta_g < \beta_n$, and $\hat{\triangle}_B \geq 0$.

□

Under conditions specified in (1.5.12) and ***Lemma 9***, $\triangle_A = \tilde{\triangle}_A$ is an optimal prize spread. Then, one must check if the designer uses good A's endowment completely.

Lemma 10. *For $\hat{\triangle}_B \geq 0$, $\tilde{\triangle}_A \in (-1, 1)$ and $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J\left(1, \hat{\triangle}_B\right)$ the designer assigns A^W, $A^L > 0$ in a reward bundle and does not waste good A's endowment.*

Proof. To prove the statement, recall properties of the designer's objective function established in ***Lemma 4***. When $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J\left(1, \hat{\triangle}_B\right)$, it must be $\frac{\partial J^+}{\partial \triangle_A}\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > 0$. First-order conditions of the designer's problem look as follows:

$$\frac{\partial L}{\partial A^W}(\cdot) = \frac{\partial J}{\partial \triangle_A}\left(\triangle_A, \hat{\triangle}_B\right) + \eta_W - \lambda$$
$$\frac{\partial L}{\partial A^L}(\cdot) = -\frac{\partial J}{\partial \triangle_A}\left(\triangle_A, \hat{\triangle}_B\right) + \eta_L - \lambda$$

Take the case of $\beta_g \geq \beta_n$ where $\tilde{\triangle}_A \in (-1, 0)$. Suppose $\lambda = \eta_L = 0$, $\eta_W > 0$ must hold in the optimum, i.e. the designer wastes a share of good A's endowment. Then, it has to be:

$$\frac{\partial L^+}{\partial A^W}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)=\frac{\partial J^+}{\partial \triangle_A}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)+\eta_W=0\Leftrightarrow\eta_W=-\frac{\partial J^+}{\partial \triangle_A}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)$$
$$\frac{\partial L^+}{\partial A^L}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)=-\frac{\partial J^+}{\partial \triangle_A}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)>0$$
$$\frac{\partial L^-}{\partial A^W}\left(\triangle_A,\,\hat{\triangle}_B\right)=\frac{\partial J^-}{\partial \triangle_A}\left(\triangle_A,\,\hat{\triangle}_B\right)+\eta_W>0\,\forall\,\triangle_A\in\left[\max\left[\triangle_A^g,\,-1\right],\,\tilde{\triangle}_A\right)$$
$$\frac{\partial L^-}{\partial A^L}\left(\triangle_A,\,\hat{\triangle}_B\right)=-\frac{\partial J^-}{\partial \triangle_A}\left(\triangle_A,\,\hat{\triangle}_B\right)<0\,\forall\,\triangle_A\in\left[\max\left[\triangle_A^g,\,-1\right],\,\tilde{\triangle}_A\right)$$

$\frac{\partial L^-}{\partial A^W}\left(\triangle_A,\,\hat{\triangle}_B\right)>0$ implies non-optimality of $A^W=0$, and the initial guess $\eta_W>0$ was incorrect. Further, assume $\lambda>0$, $\eta_{L,W}=0$, and first-order conditions become:

$$\frac{\partial L^+}{\partial A^L}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)=-\frac{\partial J^+}{\partial \triangle_A}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)-\lambda=0\Leftrightarrow\lambda=-\frac{\partial J^+}{\partial \triangle_A}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)$$
$$\frac{\partial L^+}{\partial A^W}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)=\frac{\partial J^+}{\partial \triangle_A}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)-\lambda<0$$
$$\frac{\partial L^-}{\partial A^W}\left(\triangle_A,\,\hat{\triangle}_B\right)=\frac{\partial J^-}{\partial \triangle_A}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)-\lambda>0\,\forall\,\triangle_A\in\left[\max\left[\triangle_A^g,\,-1\right],\,\tilde{\triangle}_A\right)$$
$$\frac{\partial L^+}{\partial A^L}\left(\triangle_A,\,\hat{\triangle}_B\right)=-\frac{\partial J^-}{\partial \triangle_A}\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)-\lambda<0\,\forall\,\triangle_A\in\left[\max\left[\triangle_A^g,\,-1\right],\,\tilde{\triangle}_A\right)$$

These conditions do not result in a contradiction. For $\triangle_A\in\left[\max\left[\triangle_A^g,\,-1\right],\,\tilde{\triangle}_A\right)$ the designer benefits when assigns the highest (the lowest) feasible $A^W>0$ ($A^L>0$). Under $\triangle_A\in\left[\tilde{\triangle}_A,\,1\right]$ it is optimal to choose the smallest prize spread such that both reward components are positive and the endowment is not wasted. Non-optimality of $\eta_L>0$ for the case of $\beta_g<\beta_n$ and $\tilde{\triangle}_A\in(0,\,1)$ can be shown in the same way.

□

Thus, with $\hat{\triangle}_B\geq 0$, the designer leaves a positive prize to a loser if and only if

$$\begin{cases}\alpha_g>\alpha_g^*\\ \alpha_n\in\left(0,\,\beta_n\hat{\triangle}_B\right)\end{cases}$$

When $\alpha_n>\hat{\triangle}_B\beta_n$, the designer always prefers the $\left(1,\,\hat{\triangle}_B\right)$ allocations to the $\left(\tilde{\triangle}_A,\,\hat{\triangle}_B\right)$ schedule (***Lemma 8***). Then, one must compare $J\left(1,\,\hat{\triangle}_B\right)$ and $J\left(1,\,0\right)$ directly:

$$J\left(1,\,\hat{\triangle}_B\right)>J\left(1,\,0\right)\Leftrightarrow\alpha_g>\hat{r}_1>0\tag{1.5.16}$$

Condition (1.5.16) was derived and analyzed above.

Lemma 11. *For $\beta_g>3\beta_n$ and $\hat{\triangle}_B\geq 0$ there exists $\alpha_n^*>\hat{\triangle}_B\beta_n>0$ such that $\hat{r}_1>\alpha_n+\hat{\triangle}_B\left(\beta_g-\beta_n\right)$ for any $\alpha_n>\hat{\alpha}_n$ and $\hat{r}_1<\alpha_n+\hat{\triangle}_B\left(\beta_g-\beta_n\right)$, otherwise. For $\beta_g\in(0,\,3\beta_n)$ and $\hat{\triangle}_B\geq 0$ the inequality $\hat{r}_1<\alpha_n-\hat{\triangle}_B\min\left(\beta_n-\beta_g,\,\beta_g-\beta_n\right)$ holds for any $\alpha_n\geq 0$.*

Proof. Define $f_1(\alpha_n) = \hat{r}_1 - \left(\alpha_n + \hat{\triangle}_B(\beta_g - \beta_n)\right)$ and $f_2(\alpha_n) = \hat{r}_1 - \left(\alpha_n + \hat{\triangle}_B(\beta_n - \beta_g)\right)$. First, take $\beta_g \geq \beta_n$ and solve $f_1(\alpha_n) = 0$ with respect to α_n:

$$f_1(\alpha_n) = 0 \Leftrightarrow \alpha_n = k_i,\ i = \{1,\ 2\}$$

where $k_{1,2} = \frac{\hat{\triangle}_B\left(\mp\sqrt{\beta_n(\beta_n+\beta_g)\left(8\beta_g^2-15\beta_n\beta_g+9\beta_n\right)}-\beta_n(3\beta_n-5\beta_g)\right)}{2(\beta_g-3\beta_n)}$, $k_1 < k_2$. Different outcomes emerge:

- $\beta_g \in [\beta_n,\ 3\beta_n) \Rightarrow k_2 < 0$ and $f_1(\alpha_n) < 0$ for any $\alpha_n \geq 0$
- $\beta_g > 3\beta_n \Rightarrow k_1 < 0,\ k_2 > \hat{\triangle}_B\beta_n > 0$ and $f_1(\alpha_n) \leq 0$ for $\alpha_n \in [0,\ k_2]$, $f_1(\alpha_n) > 0$ for $\alpha_n > k_2$

Taking $\alpha_n^* = k_2$ for $\beta_g > 3\beta_n$, we get the first statement of the lemma.

Second, consider the case of $\beta_g < \beta_n$ and solve $f_2(\alpha_n) = 0$ with respect to α_n:

$$f_2(\alpha_n) = 0 \Leftrightarrow \alpha_n = \frac{2\hat{\triangle}_B\beta_n(\beta_n - \beta_g)}{\beta_g - 3\beta_n} < 0\, or\, \alpha_n = -\hat{\triangle}_B\beta_n$$

Since both roots are negative, $f_2(\alpha_n)$ does not change its sign in $\alpha_n \geq 0$, and $f_2(\alpha_n) < 0$ holds for any $\alpha_n \geq 0$.

□

With ***Lemma 11***, we can specify when the designer prefers to leave good B out under $\hat{\triangle}_B \geq 0$:

$$J(1,\ 0) > J\left(1,\ \hat{\triangle}_B\right) > J\left(\tilde{\triangle}_A,\ \hat{\triangle}_B\right) \Leftrightarrow \begin{cases} \alpha_g \in \left(\alpha_n - \hat{\triangle}_B(\beta_n - \beta_g),\ \hat{r}_1\right) \\ \alpha_n > \alpha_n^*,\ \beta_g > 3\beta_n \end{cases}$$

$$J\left(1,\ \hat{\triangle}_B\right) > J(1,\ 0) > J\left(\tilde{\triangle}_A,\ \hat{\triangle}_B\right) \Leftarrow \begin{cases} \alpha_g > \hat{r}_1 \\ \alpha_n > \alpha_n^*,\ \beta_g > 3\beta_n \end{cases}$$

Next, assume $\hat{\triangle}_B < 0$. Any allocation with $\triangle_A < 0$ leads to negative winning benefits and destroys incentives to compete. The conditions, under which type-feasibility, $\tilde{\triangle}_A \in [0,\ 1]$, and $\frac{\partial J^+\left(\tilde{\triangle}_A, \hat{\triangle}_B\right)}{\partial \triangle_A} < 0$ hold simultaneously, are

$$\begin{cases} \alpha_g \in \left(\max\left\{\alpha_n + (\beta_n - \beta_g)\hat{\triangle}_B,\ 3\alpha_n\right\},\ \alpha_n\frac{\beta_g}{\beta_n}\right) \\ \alpha_n > -\beta_n\hat{\triangle}_B \\ \beta_g > 3\beta_n \end{cases}$$

Lemma 12. *For $\beta_g > 7\beta_n$ and $\hat{\triangle}_B < 0$ there exist valuation profiles such that the designer prefers to leave a positive prize for a loser.*

Proof. First, we compare $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right)$ and $J\left(1, \hat{\triangle}_B\right)$. The case of $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J\left(1, \hat{\triangle}_B\right)$ is characterized by condition (1.5.13) and roots s_1 and s_2 (***Lemma 8)*** where

$$s_1 < s_2 \Leftrightarrow \alpha_n \in \left[0, \hat{\triangle}_B \frac{(\beta_n - \beta_g)}{2}\right)$$

For $\beta_g > 3\beta_n$, this set has a non-empty intersection with $\alpha_n > -\beta_n \hat{\triangle}_B$. Also, $\alpha_n \in \left[0, \hat{\triangle}_B \frac{(\beta_n - \beta_g)}{2}\right)$ implies $\left\{\alpha_n + (\beta_n - \beta_g)\hat{\triangle}_B > 3\alpha_n\right\}$. Combining all conditions together, $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J\left(1, \hat{\triangle}_B\right)$ holds if and only if

$$\begin{cases} \alpha_g \in \left(\alpha_n + (\beta_n - \beta_g)\hat{\triangle}_B, \min\left\{\alpha_n \frac{\beta_g}{\beta_n}, s_2\right\}\right) \\ \alpha_n \in \left(-\hat{\triangle}_B \beta_n, \hat{\triangle}_B \frac{(\beta_n - \beta_g)}{2}\right) \\ \beta_g > 3\beta_n \end{cases} \tag{1.5.17}$$

Second, we investigate when $J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J(1, 0)_{\{\alpha_g > \alpha_n\}}$ holds:

$$J\left(\tilde{\triangle}_A, \hat{\triangle}_B\right) > J(1, 0)_{\{\alpha_g > \alpha_n\}} \; for \; \hat{\triangle}_B < 0 \Leftrightarrow \alpha_g \in [0, \bar{r}_1)$$

where $\bar{r}_1$ was defined in ***Lemma 9.*** This condition delivers a non-empty intersection with (1.5.17) if and only if

$$\begin{cases} \alpha_n \in \left(-\hat{\triangle}_B \beta_n, \min\left\{\frac{2\hat{\triangle}_B \beta_n (\beta_n - \beta_g)}{\beta_g - 3\beta_n}, \frac{\hat{\triangle}_B (\beta_n - \beta_g)}{2}\right\}\right) \\ \beta_g > 7\beta_n \end{cases}$$

Lemma 11 illustrates that the designer indeed assigns $A^L > 0$ in the bundle.

□

Using the proof of ***Lemma 12***, one can also show when the designer prefers to leave good B out and run a one-dimensional contest over item A.

□

Theorem 1.1. *For any valuation profile $r \in R$ the designer uses goods' endowments completely and either*

- *Leaves a positive losing prize at least in one dimension or*
- *Gives both items to a winner.*

Proof. Let $r = \{\alpha_g, \alpha_n, \beta_g, \beta_n\}$. A complete characterization of the designer's problem looks as follows:

$$L\left(A^W, A^L, B^W, B^L, \{\eta_i\}_{i=W}^L, \{\kappa_i\}_{i=W}^L, \{\lambda_i\}_{i=B}^\theta\right) = J\left(A^W, A^L, B^W, B^L\right) + \\ +\eta_W A^W + \eta_L A^L + \kappa_W B^W + \kappa_L B^L + \lambda_A\left(1 - A^W - A^L\right) + \lambda_B\left(1 - A^W - A^L\right)$$

The designer's objective function can be rewritten in terms of prize spreads:

$$J\left(A^W, A^L, B^W, B^L\right) \equiv J\left(\triangle_A, \triangle_B\right)$$

We call player i, $i = \{1, 2\}$ type g if and only if $t_i(1, 1) > t_{-i}(1, 1)$.

Lemma 1. *The designer's problem has no interior solution.*

Proof. To prove the statement, compute the hessian of $J(\triangle_A, \triangle_B)$:

$$H = \begin{bmatrix} \frac{\triangle_B^2(\beta_n\alpha_g - \beta_g\alpha_n)^2}{t_g^2} & -\frac{\triangle_A\triangle_B(\beta_n\alpha_g - \beta_g\alpha_n)^2}{(\alpha_g\triangle_A + \beta_g\triangle_B)^3} \\ -\frac{\triangle_A\triangle_B(\beta_n\alpha_g - \beta_g\alpha_n)^2}{(\alpha_g\triangle_A + \beta_g\triangle_B)^3} & \frac{\triangle_A^2(\beta_n\alpha_g - \beta_g\alpha_n)^2}{t_g^2} \end{bmatrix}$$

Since H is not negative definite, any prize allocation that satisfies at least one first order condition corresponds to an interior minimum.

□

Corollary 1. $\frac{\partial J}{\partial \triangle_A}(\cdot)$ *and* $\frac{\partial J}{\partial \triangle_B}(\cdot)$ *increase in* $\triangle_A$ *and* $\triangle_B$*, respectively.*

Since the designer can change the prize allocation in both dimensions, one does not need to consider reward schedules $(\triangle_A, 0)$ and $(0, \triangle_B)$, where one good is left out, separately. Fixing the prize scheme in one of the dimensions, define $\tilde{\triangle}$'s in items A and B such that contestants' types equalize:

$$t_g(\cdot) = t_n(\cdot) \Leftrightarrow \tilde{\triangle}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n}\triangle_B \; or \; \tilde{\triangle}_B = \frac{\alpha_n - \alpha_g}{\beta_g - \beta_n}\triangle_A$$

and the types are well-defined if and only if

$$t_j\left(\triangle_A, \hat{\triangle}_B\right) \geq 0 \Leftrightarrow \triangle_A^j \geq -\frac{\beta_j}{\alpha_j}\triangle_B \; or \; \triangle_B^j \geq -\frac{\alpha_j}{\beta_j}\triangle_A, \; j = \{g, n\}$$

It is easy to prove that $\tilde{\triangle}_A$ ($\tilde{\triangle}_B$) cannot be feasible for $\alpha_g < \alpha_n$ ($\beta_g < \beta_n$):

$$\tilde{\triangle}_A \leq 1 \Leftrightarrow \tfrac{\beta_n - \beta_g}{\alpha_g - \alpha_n}\triangle_B \leq 1 \Leftrightarrow \alpha_g \leq \alpha_n + (\beta_n - \beta_g)\triangle_B$$
$$\tilde{\triangle}_B \leq 1 \Leftrightarrow \tfrac{\alpha_n - \alpha_g}{\beta_g - \beta_n}\triangle_A \leq 1 \Leftrightarrow \beta_g \leq \beta_n + (\alpha_n - \alpha_g)\triangle_A$$

Both inequalities contradict the definition of type g provided above. Thus, a necessary condition for feasibility of $\triangle_A = \tilde{\triangle}_A$ ($\triangle_B = \tilde{\triangle}_B$) is $\alpha_g > \alpha_n$ ($\beta_g > \beta_n$).

Definition. $\tilde{\triangle}_k$, $k = \{A, B\}$ is type-feasible if and only if $\max\{\triangle_k^g, \triangle_k^n\} < \tilde{\triangle}_k$.

The proof of Proposition 1.2 (***Lemma 2***) states that $\frac{\partial J}{\partial \triangle_A}(\cdot)$ has a jump point at $\triangle_A = \tilde{\triangle}_A$ for any $\alpha_g \neq \alpha_n$. A similar result holds for $\frac{\partial J}{\partial \triangle_B}(\cdot)$ and $\beta_g \neq \beta_n$.

Lemma 2. *$\frac{\partial J}{\partial \triangle_B}(\cdot)$ has a jump point at $\triangle_B = \tilde{\triangle}_B$ for any $\beta_g \neq \beta_n$.*

Proof. The limits of $\frac{\partial J}{\partial \triangle_B}(\cdot)$ from the left and from the right of $\triangle_B = \tilde{\triangle}_B$ are

$$\lim_{\triangle_B \to -\tilde{\triangle}_B} \frac{\partial J}{\partial \triangle_B}(\cdot) = \frac{3\beta_g - \beta_n}{2}$$
$$\lim_{\triangle_B \to +\tilde{\triangle}_B} \frac{\partial J}{\partial \triangle_B}(\cdot) = \frac{3\beta_n - \beta_g}{2}$$

These values coincide if and only if $\beta_g = \beta_n$. Thus, $\frac{\partial J}{\partial \triangle_B}(\cdot)$ has a jump point at $\triangle_B = \tilde{\triangle}_B$ for any $\beta_g \neq \beta_n$.

□

We treat $\frac{\partial J}{\partial \triangle_A}(\cdot)$ and $\frac{\partial J}{\partial \triangle_B}(\cdot)$ as right-continuous function and denote their derivatives to the left and to the right of $\tilde{\triangle}$'s with the "–" and "+" signs, respectively.

Lemma 3. *For any valuation profile such that $\tilde{\triangle}_A$ ($\tilde{\triangle}_B$) is not type-feasible $\frac{\partial J}{\partial \triangle_A}(\max\{\triangle_A^g, \triangle_A^n\}, \triangle_B) > 0$ ($\frac{\partial J}{\partial \triangle_B}(\triangle_A, \max\{\triangle_B^g, \triangle_B^n\}) > 0$).*

Proof. The proof of Proposition 1.2 (***Lemma 3***) shows that there is no type-feasibility in dimension A if and only if $\max\{\triangle_A^g, \triangle_A^n\} = \triangle_A^n$. One can apply similar arguments and prove that $\max\{\triangle_B^g, \triangle_B^n\} = \triangle_B^n$ in the absence of type-feasibility over good B.

Next, evaluate $\frac{\partial J}{\partial \triangle_A}(\triangle_A^n, \triangle_B)$ and $\frac{\partial J}{\partial \triangle_B}(\triangle_A, \triangle_B^n)$:

$$\frac{\partial J}{\partial \triangle_A}(\triangle_A^n, \triangle_B) = \frac{\alpha_n}{2} > 0$$
$$\frac{\partial J}{\partial \triangle_B}(\triangle_A, \triangle_B^n) = \frac{\beta_n}{2} > 0$$

and the statement of the lemma follows.

□

Lemma 4. *For any valuation profile such that $\tilde{\triangle}_k$, $k = \{g, n\}$ is type-feasible the objective function $J(\cdot)$ strictly increases in $\triangle_k$ for $\triangle_k \in \left[\triangle_k^g, \tilde{\triangle}_k\right)$. $J(\cdot)$ strictly decreases or is non-monotone in $\triangle_k$ for $\triangle_k \in \left[\tilde{\triangle}_k, 1\right]$ if and only if $\frac{\partial J}{\partial \triangle_k}^+\left(\tilde{\triangle}_k\right) < 0$; otherwise, $J(\cdot)$ strictly increases in $\triangle_k$ for $\triangle_k \in \left[\tilde{\triangle}_k, 1\right]$.*

Proof. The proof of Proposition 1.2 (***Lemma 4***) characterizes the case of type-feasibility in dimension A $(k = A)$. The same set of arguments can be used to prove the lemma for $k = B$ when $\beta_g > 3\beta_n$ is necessary to support non-monotonicity of $J(\cdot)$ in $\triangle_k$. □

Enough heterogeneity in contestants' valuations over a particular dimension ($\alpha_g > 3\alpha_n$ or / and $\beta_g > 3\beta_n$) is necessary (but not sufficient) to make a positive losing prize optimal. Given ***Lemma 1*** and ***Lemma 4***, the optimal reward schedule is a corner solution of the designer's problem. Then, the following alternatives must be compared directly when type-feasibility holds:

$$(1,\, 1)\ vs.\ \left(1,\, \max\left\{\min\left\{\tilde{\triangle}_B,\, 1\right\},\, -1\right\}\right)\ vs.\ \left(\max\left\{\min\left\{\tilde{\triangle}_A,\, 1\right\},\, -1\right\},\, 1\right)\ vs.$$
$$\left(-1,\, \max\left\{\min\left\{\tilde{\triangle}_B,\, 1\right\},\, -1\right\}\right) ...$$

A single allocation that cannot feature positive losing prizes is the $(1,\, 1)$ schedule. To prove the optimality of $\triangle_A \neq 1$ or / and $\triangle_B \neq 1$, it is sufficient to show when $(1,\, 1)$ is dominated by at least one reward schedule with $\triangle_A < 1$ or / and $\triangle_B < 1$. Take Proposition 2 and fix $\hat{\triangle}_B = 1$. Then, there must exist valuation profiles such that the designer prefers to use both goods and leaves a positive losing prize in dimension A. With the details from the proof of Proposition 1.2, we construct an example. Assume $\beta_g > \beta_n$ and $\frac{\partial J}{\partial \triangle_B}^{+}\left(\tilde{\triangle}_B\right) > 0 \Leftrightarrow \beta_g \in (\beta_n,\, 3\beta_n)$. Then, ***Corollary 1, Lemma 3*** and ***Lemma 4*** imply that $\frac{\partial J}{\partial \triangle_B}(\cdot) > 0$ holds for any feasible prize allocation, and $\triangle_B = 1$ is optimal. Next, take $\hat{\triangle}_B = 1$ and consider the case when $\tilde{\triangle}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n}$ is type-feasible. The designer prefers the $\left(\tilde{\triangle}_A,\, 1\right)$ allocation to the $(1,\, 1)$ schedule if and only if

$$\begin{cases} \alpha_g > \max\left\{\alpha_n + (\beta_g - \beta_n)\,,\, 3\alpha_n,\, \alpha_n \frac{\beta_g}{\beta_n},\, \alpha_g^*\right\} \\ \alpha_n \in (0,\, \beta_n)\,,\, \beta_g \in (\beta_n,\, 3\beta_n) \end{cases}$$

where all conditions and α_g^* were defined in the proof of Proposition 1.2. One can construct examples for other valuation profiles similarly. To show that the designer uses goods' endowments completely, the proof of Proposition 1.2 (***Lemma 10***) applies.

□

Theorem 1.1. *There exists a non-empty subset of R, R^b, such that for any $\{\alpha_g,\, \alpha_n,\, \beta_g,\, \beta_n\} \in R^b$ the designer prefers one contest with bundled prizes to two separate contests with unbundled prizes.*

Proof. To show the existence of $R^b \neq \emptyset$ we refer to the proofs of Proposition 1.2 and Theorem 1.1. First, consider the case when the designer prefers to assign a positive losing prize in good A ($\triangle_A = \max\left\{\tilde{\triangle}_A,\, -1\right\}$) and gives item B to a winner ($\triangle_B = 1$, and $\beta_g \in (\beta_n,\, 3\beta_n)$ is assumed):

1. $\tilde{\triangle}_A \leq -1 \Leftrightarrow \alpha_g < \alpha_n - \beta_n + \beta_g$, i.e. $\tilde{\triangle}_A$ is not feasible.

   The optimality of $(-1,\, 1)$ requires

$$\begin{cases} \alpha_g \in \left(\max\left\{r_1,\, 3\alpha_n\right\},\, \alpha_n - (\beta_n - \beta_g)\, \hat{\triangle}_B\right) \\ \alpha_n \in [0,\, q_1)\,,\, \beta_g \in (\beta_n,\, 3\beta_n) \end{cases} \tag{1.5.18}$$

   where all roots and conditions were defined in the proof of Proposition 1.2 ($\hat{\triangle}_B = 1$ is taken). The $(-1,\, 1)$ schedule dominates two separate contests over dimensions A and B if and only if

$$J\,(-1,\, 1) > J\,(1,\, 0)_{\{\alpha_g > \alpha_n\}} + J\,(0,\, 1)_{\{\beta_g > \beta_n\}} \Leftrightarrow$$
$$\left(\beta_n^2 + 2\beta_g\alpha_n\right)\alpha_g^2 - 2\beta_g\alpha_n\left(\beta_g + \beta_n - \alpha_n\right)\alpha_g - \beta_g^2\alpha_n^2 > 0$$

   The underlying square equation always has two real roots, h_1 and h_2, $h_1 < 0 < h_2$. Then, $J\,(-1,\, 1) > J\,(1,\, 0)_{\{\alpha_g > \alpha_n\}} + J\,(0,\, 1)_{\{\beta_g > \beta_n\}}$ holds if and only if $\alpha_g > h_2$, and this set has a non-empty intersection with (1.5.18) if and only if $h_2 < \alpha_n - \beta_n + \beta_g$.

   Lemma 1. *For $\tilde{\triangle}_A \leq -1$ $\beta_g > \beta_n$ there exists $\alpha_n^h \in (0,\, \beta_n)$ such that $h_2 < \alpha_n - \beta_n + \beta_g$ for any $\alpha_n \in \left(0,\, \alpha_n^h\right)$.*

   Proof. Define $f^h\,(\alpha_n) = h_2 - \alpha_n - (\beta_g - \beta_n)$. Solving $f^h\,(\alpha_n) = 0$ with respect to α_n delivers:

$$f^{\varphi}\,(\alpha_n) = 0 \Leftrightarrow \alpha_n = \alpha_n^i,\ i = \{1,\, ...,\, 4\}$$
$$\alpha_n^1 = \beta_n,\ \alpha_n^2 = -\tfrac{\beta_n^2}{2\beta_g}$$
$$\alpha_n^j = \tfrac{-3\beta_g^2 - \beta_n^2 + 4\beta_g\beta_n \mp (\beta_g - \beta_n)\sqrt{(\beta_g + \beta_n)(9\beta_g + \beta_n)}}{8\beta_g},\ j = \{3,\, 4\}$$
$$\alpha_n^3 < 0 < \alpha_n^4 < \beta_n$$

   where $f^h\,(\alpha_n) < 0$ for $\alpha_n \in (0,\, \alpha_n^4)$ and $f^h\,(\alpha_n) > 0$ for $\alpha_n \in (\alpha_n^4,\, \beta_n)$. Taking $\alpha_n^h = \alpha_n^4$ gives the statement of the lemma.

   □

Thus, the designer prefers the $(-1,\,1)$ allocation to two separate contests if and only if

$$r \in R_1^b = \begin{cases} \alpha_g \in (\max\{r_1,\, 3\alpha_n,\, h_2\},\, \alpha_n - \beta_n + \beta_g) \\ \alpha_n \in \left(0,\, \min\left\{q_1,\, \alpha_n^h\right\}\right),\, \beta_g \in (\beta_n,\, 3\beta_n) \end{cases}$$

2. $\tilde{\triangle}_A \in (-1,\,1) \Leftrightarrow \alpha_g > \alpha_n + \beta_g - \beta_n$ for $\beta_g \in (\beta_n,\, 3\beta_n)$, i.e. $\tilde{\triangle}_A$ is feasible.

   With two-dimensional rewards, the designer assigns a positive losing prize in dimension A if and only if

$$\begin{cases} \alpha_g > \max\left\{\alpha_n + \beta_g - \beta_n,\, 3\alpha_n,\, \alpha_n \frac{\beta_g}{\beta_n},\, \alpha_g^*\right\} \\ \alpha_n \in (0,\, \beta_n),\, \beta_g \in (\beta_n,\, 3\beta_n) \end{cases} \tag{1.5.19}$$

   The $\left(\tilde{\triangle}_A,\, 1\right)$ bundle dominates two separate contests with single-item prizes if and only if:

$$J\left(\tilde{\triangle}_A,\, 1\right) > J(1,\,0)_{\{\alpha_g > \alpha_n\}} + J(0,\,1)_{\{\beta_g > \beta_n\}} \Leftrightarrow \gamma_1 \alpha_g^2 + \gamma_2 \alpha_g + \gamma_3 > 0$$
$$\gamma_1 = -\beta_n^2 + \beta_g \beta_n - \beta_g \alpha_n,\, \gamma_2 = \alpha_n\left(-2\beta_g^2 + \beta_g \beta_n + \beta_n^2\right)$$
$$\gamma_3 = \beta_g^2 \alpha_n^3$$
$$\gamma_1 > 0 \Leftrightarrow \alpha_n < \tfrac{\beta_n(\beta_g - \beta_n)}{\beta_g},\, \gamma_2 < 0\, \forall\, \beta_g > \beta_n$$

   The corresponding square equation always has two real roots, r_γ^1 and r_γ^2, $r_\gamma^1 < r_\gamma^2$. Take the case of $\alpha_n \in \left(0,\, \frac{\beta_n(\beta_g - \beta_n)}{\beta_g}\right)$ when $\gamma_1 > 0$. This implies $r_\gamma^1 > 0$, and the bundle is optimal if and only if $\alpha_g \in \left(0,\, r_\gamma^1\right) \cup \left(r_\gamma^2,\, \infty\right)$. Then, a sufficient condition to guarantee the optimality of the $\left(\tilde{\triangle}_A,\, 1\right)$ allocation over two separate contests is

$$r \in R_2^b = \begin{cases} \alpha_g > \max\left\{\alpha_n + \beta_g - \beta_n,\, 3\alpha_n,\, \alpha_n \frac{\beta_g}{\beta_n},\, \alpha_g^*,\, r_\gamma^2\right\} \\ \alpha_n \in \left(0,\, \frac{\beta_n(\beta_g - \beta_n)}{\beta_g}\right),\, \beta_g \in (\beta_n,\, 3\beta_n) \end{cases}$$

Next, we show when the $(1,\,1)$ bundle dominates two simultaneous competitions over dimensions A and B. In the beginning, assume $\alpha_g > \alpha_n$ and $\beta_g > \beta_n$. One contest with bundled prizes is preferred to two separate competitions if and only if

$$J(1,\,1) > J(1,\,0)_{\{\alpha_g > \alpha_n\}} + J(0,\,1)_{\{\beta_g > \beta_n\}} \Leftrightarrow$$
$$\Leftrightarrow -\tfrac{(\beta_g \alpha_n - \beta_n \alpha_g)^2}{2\beta_g \alpha_g (\alpha_g + \beta_g)} > 0$$

This condition never holds for $\beta_g,\, \alpha_g > 0$. Thus, if type g values both goods more than his opponent, the "Winner-Takes-All" bundle never dominates two simultaneous competitions.

Now, we analyze the case of $\alpha_g > \alpha_n$ and $\beta_g < \beta_n$. To support the optimality of $\triangle_A = 1$, we impose $\alpha_g \in (\alpha_n, 3\alpha_n)$ (see the proof of Theorem 1.1). The "Winner-Takes-All" bundle dominates other feasible two-dimensional allocations if and only if

$$\begin{cases} \alpha_g \in (\alpha_n + \beta_n - \beta_g, \, 3\alpha_n), \, \alpha_n > \frac{\beta_n - \beta_g}{2} \\ \beta_g \in [0, \, \beta_n) \end{cases} \tag{1.5.20}$$

The designer prefers the (1, 1) allocation to two simultaneous contests with single-item rewards if and only if

$$J(1,\,1) > J(1,\,0)_{\{\alpha_g > \alpha_n\}} + J(0,\,1)_{\{\beta_g < \beta_n\}} \Leftrightarrow \eta_1 \alpha_g^2 + \eta_2 \alpha_g + \eta_3 > 0$$
$$\eta_1 = -\beta_g^2 - \beta_g \beta_n + \beta_n^2, \, \eta_2 = (\beta_n - \beta_g)(\beta_g + \beta_n)^2 + 2\alpha_n \beta_n^2 > 0$$
$$\eta_3 = -\alpha_n^2 \beta_n \beta_g < 0$$
$$\eta_1 > 0 \Leftrightarrow \beta_g \in \left[0, \, \bar{\beta}_g\right), \, \bar{\beta}_g = \frac{\sqrt{5}-1}{2}\beta_n < \beta_n$$

For $\beta_g < \beta_n$, the underlying square equation always has two real roots, r_η^1 and r_η^2. Take the case of $\beta_g \in \left[\frac{\beta_n}{3}, \, \bar{\beta}_g\right)$ when $\eta_1 > 0$. Then, $r_\eta^1 < 0 < r_\eta^2$ holds, and the (1, 1) bundle dominates two single-item contests if and only if $\alpha_g > r_\eta^2$. The condition $r_\eta^2 < \alpha_n + \beta_n - \beta_g$ holds for any $\alpha_n > 0$ and $\beta_g < \beta_n$. Then, $\alpha_g \in (\alpha_n + \beta_n - \beta_g, \, 3\alpha_n)$ implies $\alpha_g > r_\eta^2$, and the designer prefers the "Winner-Takes-All" bundle to other alternative reward schedules if and only if

$$r \in R_3^b = \begin{cases} \alpha_g \in (\alpha_n + \beta_n - \beta_g, \, 3\alpha_n), \, \alpha_n > \frac{\beta_n - \beta_g}{2} \\ \beta_g \in [0, \, \beta_n) \end{cases}$$

Taking a union of R_1^b, R_2^b, and R_3^b, we define a non-empty subset of R^b:

$$\left(R_1^b \cup R_2^b \cup R_3^b\right) \subseteq R^b$$

□

Proposition 1.4. *For* $\min\{t_1^g, t_2^g\} > 0$ *there exists a unique non-trivial equilibrium in monotonically increasing strategies such that:*

- *At least one type places an atom at zero:* $\exists i = \{1, 2\}, j = \{g, n\} : G_i\left(t_i^j, 0\right) > 0$;
- *There is no* $e > 0$ *played with a positive probability;*
- *Supports of* $G_i\left(t_i^g, e\right)$, $i = \{1, 2\}$ *have the same supremum:* $\sup\left(s_1^g\right) = \sup\left(s_2^g\right)$;
- *Supports of* $G_i\left(t_i^n, e\right)$, $i = \{1, 2\}$ *have the same infimum, and it is equal to zero:* $\inf\left(s_1^n\right) = \inf\left(s_2^n\right) = 0$.

Proof. To show the existence, I construct an equilibrium. Given properties derived in Siegel (2014), the equilibrium of this game must look like a partition where various types randomize on specific intervals. Another result states that this partition can be obtained using a "top-down" approach.

Let $U_{i,l}^L$ correspond to the losing benefit of contestant i with type l, $l = \{g, n\}$. Assume $t_i^g > t_j^g$, $i, j = \{1, 2\}$. First, I show that $t_j^g \leq 0$ never supports the non-trivial equilibrium.

Lemma 1. *For* $t_j^g \leq 0$ *there does not exist a non-trivial equilibrium.*

Proof. Assume $t_j^g \leq 0$ chooses $e_j^g > 0$ in equilibrium. If he wins, the payoff becomes:
$$\pi_{j,g}^W\left(e_j^g\right) = t_j^g + U_{j,g}^L - e_j^g$$
Bidding $e_j^g = 0$, the contestant gets $\pi_{j,g}^W(0) = t_j^g + U_{j,g}^L$ ($\pi_{j,g}^L(0) = U_{j,g}^L$) if wins (loses). Then $\pi_{j,g}^L(0) \geq \pi_{j,g}^W(0) > \pi_{j,g}^W\left(e_j^g\right)$ for any $e_j^g > 0$, $t_j^g \leq 0$, and $e_j^g > 0$ cannot be in equilibrium. As a result, $t_j^g \leq 0$ must place $p^0\left(t_j^g\right) = 1$ at $e = 0$.

□

Next, consider $t_j^g > 0$ and characterize the top-interval of the partition where t_i^g and t_j^g play against each other. Take two effort choices, $0 < e_i^g < \tilde{e}_i^g$, corresponding to best responses of t_i^g and generating the same equilibrium payoff. Then it must be:

$$\begin{aligned}
&P\left(t_j^n\right)\left(t_i^g + U_{i,g}^L\right) + P\left(t_j^g\right)\left(t_i^g G_j\left(t_j^g, \tilde{e}_i^g\right) + U_{i,g}^L\right) - \tilde{e}_i^g = \\
&= P\left(t_j^n\right)\left(t_i^g + U_{i,g}^L\right) + P\left(t_j^g\right)\left(t_i^g G_j\left(t_j^g, e_i^g\right) + U_{i,g}^L\right) - e_i^g
\end{aligned}$$

The first element of the sum reflects the case when t_i^g plays against t_j^n and wins with certainty because t_j^n never bids in the top interval. Matched against t_j^g, t_i^g succeeds with probability $G_j\left(t_j^g, e\right)$ (the second term of the sum). Simplification delivers
$$\frac{G_j\left(t_j^g, \tilde{e}_i^g\right) - G_j\left(t_j^g, e_i^g\right)}{\tilde{e}_i^g - e_i^g} = \frac{1}{P\left(t_j^g\right) t_i^g}$$

where the right-hand side is constant. Taking $\tilde{e}_i^g - e_i^g \to 0$ brings $g_i\left(t_i^g,\, e\right) = \frac{1}{P\left(t_i^g\right)t_j^g}$. Similarly, one can obtain $g_j\left(t_j^g,\, e\right) = \frac{1}{P\left(t_j^g\right)t_i^g}$.

Type t_i^g $\left(t_j^g\right)$ exhausts his bidding probability in the top-interval iff its length $\left(L_{top}\right)$ is equal to $P\left(t_i^g\right)t_j^g$ $\left(P\left(t_j^g\right)t_i^g\right)$. Since $t_j^g < t_i^g$ by assumption, it must be $L_{top} = P\left(t_i^g\right)t_j^g$; otherwise, there is a contradiction:

$$L_{top} = P\left(t_j^g\right)t_i^g \Rightarrow G_i\left(t_i^g,\, P\left(t_j^g\right)t_i^g\right) = \frac{t_i^g}{t_j^g} > 1$$

Thus, t_i^g exhausts his bidding probability, and t_j^g has $\left\{1 - \frac{t_j^g}{t_i^g}\right\}$ to expend in the game against t_i^n.

Moving downward, take the interval where t_j^g and t_i^n play. Several possibilities emerge:

1. $t_i^n > 0$.

   Following similar steps, one can show:

$$g_i\left(t_i^n,\, e\right) = \frac{1}{P\left(t_i^n\right)t_j^g},\; g_j\left(t_j^g,\, e\right) = \frac{1}{P\left(t_j^g\right)t_i^n}$$

   Types' bidding probabilities exhaust in the intervals of length $L_m^i = P\left(t_i^n\right)t_j^g$ for t_i^n and $L_m^j = P\left(t_j^g\right)\frac{\left(t_i^g - t_j^g\right)t_i^n}{t_i^g}$ for t_j^g (t_j^g cannot expend more than $\left\{1 - \frac{t_j^g}{t_i^g}\right\}$). Then $L_m^i > L_m^j$ holds iff:[88]

$$L_m^i > L_m^j \Leftrightarrow P\left(t_j^g\right) < \frac{t_i^g t_j^g}{t_i^n\left(t_i^g - t_j^g\right) + t_i^g t_j^g} \equiv \tilde{T},\, \tilde{T} \in [0,\, 1)$$

   (a) $P\left(t_j^g\right) \in \left[0,\, \tilde{T}\right) \Rightarrow L_m = L_m^j$. This implies that type t_j^g exhausts his bidding probability in L_m and t_i^n expends $\left\{\frac{P\left(t_j^g\right)\left(t_i^g - t_j^g\right)t_i^n}{t_i^g t_j^g P\left(t_i^n\right)}\right\}$. As a result, t_i^n can compete against t_j^n in the lowest segment of the equilibrium partition that has length L_b.

   In the beginning, take the case of $t_j^n > 0$. Then the strategies of n types become:

$$G_i\left(t_i^n,\, e\right) = \frac{e}{P\left(t_i^n\right)t_j^n},\; G_j\left(t_j^n,\, e\right) = \frac{e}{P\left(t_j^n\right)t_i^n}$$

   t_j^n and t_i^n exhaust their bidding probabilities in intervals of length $L_b^j = P\left(t_j^n\right)t_i^n$ and $L_b^i = t_j^n\left(\frac{P\left(t_i^n\right)t_i^g t_j^g - P\left(t_j^g\right)\left(t_i^g - t_j^g\right)t_i^n}{t_i^g t_j^g}\right)$, respectively:

$$L_b^i < L_b^j \Leftrightarrow P\left(t_j^g\right) > \frac{t_i^g t_j^g\left(t_j^n - t_i^n\right)}{t_j^n t_i^n\left(t_i^g - t_j^g\right) + t_i^g t_j^g\left(t_j^n - t_i^n\right)} \equiv \hat{T},\, \hat{T} \in (0,\, 1)\,,\, \hat{T} < \tilde{T}$$

[88] I use $P\left(t_i^n\right) = P\left(t_j^n\right) = 1 - P\left(t_j^g\right)$ to simplify the expression.

Consider these options separately:

- $L_b^i < L_b^j \Rightarrow L_b = L_b^i$, and t_i^n exhausts his bidding probability on the interval.
  Lemma 2. $\inf\left(s_1^n\right) = \inf\left(s_2^n\right) = 0$ *in equilibrium.*

  Proof. The argument is similar to one provided in the proof of Proposition 1.1, ***Lemma 2***.
  As t_j^n still has some bidding probability left, he places $p_1^0\left(t_j^n\right)$ at $e = 0$, $p_1^0\left(t_j^n\right) = 1 - \frac{t_j^n}{P\left(t_j^n\right)t_i^n} t_j^n \left(\frac{P\left(t_i^n\right)t_i^g t_j^g - P\left(t_j^g\right)\left(t_i^g - t_j^g\right)t_i^n}{t_i^g t_j^g}\right)$.

  $\square$

- $L_b^i > L_b^j \Rightarrow L_b = L_b^j$, and t_j^n exhausts his bidding probability in the interval. ***Lemma 2*** holds, and t_i^n must place $p_1^0\left(t_i^n\right) = \left(1 - \frac{P\left(t_j^g\right)\left(t_i^g - t_j^g\right)t_i^n t_j^n + P\left(t_j^n\right)t_i^n t_i^g t_j^g}{t_i^g t_j^g t_j^n P\left(t_i^n\right)}\right)$ at $e = 0$.

To sum up, for $P\left(t_j^g\right) \in \left[0, \tilde{T}\right)$, $t_j^n > 0$ the equilibrium partition T_i, $i = \{1, 2, 3\}$ and contestants' strategies can be characterized as follows:

- $T_1 = L_b$:
  - $P\left(t_j^g\right) \in \left(\max\left\{0, \hat{T}\right\}, \tilde{T}\right)$: with probability $\left\{1 - \frac{P\left(t_j^g\right)\left(t_i^g - t_j^g\right)t_i^n}{t_i^g t_j^g P\left(t_i^n\right)}\right\}$ t_i^n randomizes uniformly on $[0, T_1]$; t_j^n uses a strategy including uniform randomization on $(0, T_1]$ and the atom $p_1^0\left(t_j^n\right) \in (0, 1)$ placed at $e = 0$;
  - $P\left(t_j^g\right) \in \left[0, \max\left\{0, \hat{T}\right\}\right]$: t_j^n randomizes uniformly on $[0, T_1]$ with probability 1; t_i^n uses a strategy including uniform randomization on $(0, T_1]$ and the atom $p_1^0\left(t_i^n\right) \in (0, 1)$ placed at $e = 0$;
- $T_2 = T_1 + L_m$: t_i^n and t_j^g randomize uniformly on $(T_1, T_2]$ with probabilities $\left\{\frac{P\left(t_j^g\right)\left(t_i^g - t_j^g\right)t_i^n}{t_i^g t_j^g P\left(t_i^n\right)}\right\}$ and $\left\{1 - \frac{t_j^g}{t_i^g}\right\}$, respectively;
- $T_3 = T_2 + L_{top}$: t_i^g and t_j^g randomize uniformly on $(T_2, T_3]$ with probabilities 1 and $\left\{\frac{t_j^g}{t_i^g}\right\}$, respectively.

Further suppose $t_j^n \leq 0$. In this case type t_j^n places the atom $p_2^0\left(t_j^n\right) = 1$ at $e = 0$ (the argument is similar to ***Lemma 1***). ***Lemma 3*** characterizes the strategy of t_i^n:

Lemma 3. *For* $P\left(t_j^g\right) \in \left[0, \tilde{T}\right)$, $t_j^n \leq 0$ *type* n *of contestant* i $\left(t_i^n\right)$ *places* $p_2^0\left(t_i^n\right) = 1 - \frac{P\left(t_j^g\right)\left(t_i^g - t_j^g\right)t_i^n}{t_i^g t_j^g P\left(t_i^n\right)} \in (0, 1)$ *at* $e = 0$.

Proof. When $t_j^n \leq 0$, type t_j^n chooses $e = 0$ with probability 1 (the argument is similar to ***Lemma 1***). Suppose there exists $y > 0$ such that t_j^g never bids in $[0, y]$, but t_i^n randomizes uniformly on $[0, y]$ when plays against t_j^n in equilibrium. For any $e_i^n \in (0, y]$ type t_i^n wins with certainty and gets $\pi_{i,n}^W\left(e_i^n\right) = t_i^n + U_W^L - e_i^n$. Take e_i^n, $\tilde{e}_i^n \in (0, y]$,

$e_i^n > \tilde{e}_i^n$. Then $\pi_{i,n}^W(\tilde{e}_i^n) > \pi_{i,n}^W(e_i^n)$, and $\tilde{e}_i^n$ dominates e_i^n. that cannot be in equilibrium. As a result, type t_i^n must place the atom at $e \geq 0$ when plays against $t_j^n \leq 0$.

Further assume t_i^n places the atom at $e = q > 0$ in equilibrium and wins against t_j^n with certainty. Take $\varepsilon > 0$ small enough, $\varepsilon \in (0, q)$. Then $\pi_{i,n}^W(q - \varepsilon) > \pi_{i,n}^W(q)$ holds for any $\varepsilon > 0$, and there exists a profitable deviation. As a result, for $t_j^n \leq 0$ type t_i^n never places the atom at $e > 0$ in equilibrium. Since there is no other type of contestant j to compete with, t_i^n must expend his bidding probability left by allocating $p_2^0(t_i^n) = 1 - \frac{P(t_j^g)(t_i^g - t_j^g)t_i^n}{t_i^g t_j^g P(t_i^n)} \in (0, 1)$ to $e = 0$.

□

Summing up, for $P\left(t_j^g\right) \in \left[0, \tilde{T}\right)$, $t_j^n \leq 0$ the equilibrium partition M_i, $i = \{1, 2\}$ and contestants' strategies look as follows:

- $M_1 = L_m$: t_j^n and t_i^n place $p_2^0\left(t_j^n\right) = 1$ and $p_2^0(t_i^n) \in (0, 1)$ at $e = 0$, respectively; t_i^n and t_j^g randomize uniformly on $(0, M_1]$ with corresponding probabilities $\{1 - p_2^0(t_i^n)\}$ and $\left\{1 - \frac{t_j^g}{t_i^g}\right\}$;
- $M_2 = M_1 + L_{top}$: t_i^g and t_j^g randomize uniformly on $(M_1, M_2]$ with probabilities 1 and $\left\{\frac{t_j^g}{t_i^g}\right\}$, respectively.

(b) $P\left(t_j^g\right) \geq \tilde{T} \Rightarrow L_m = L_m^i$. In this case type t_j^g still has a positive bidding probability left, but t_i^n exhausts his strategy in the lowest interval of the equilibrium partition. Since there is no other type of contestant i to compete with, t_j^g must place $p_1^0\left(t_j^g\right) = 1 - \frac{t_j^g\left(t_i^g P(t_i^n) + t_i^n P(t_j^g)\right)}{P(t_j^g) t_i^n t_i^g} \in (0, 1)$ at $e = 0$. Then the equilibrium partition is characterized by thresholds $D_1 = P(t_i^n)\, t_j^g$ and $D_2 = P(t_i^g)\, t_j^g + D_1 = t_j^g$.

Lemma 4. *For $P\left(t_j^g\right) \geq \tilde{T}$ type n of contestant j places $p_3^0\left(t_j^n\right) = 1$ at $e = 0$.*

Proof. Suppose t_j^n randomizes in $(0, h) \subseteq (0, D_1)$ in equilibrium. Then t_j^n always loses against t_i^g (t_i^g bids on $[D_1, D_2]$), but can win with a positive probability if he faces t_i^n. Take e_j^n, $\tilde{e}_j^n \in (0, h)$; in equilibrium t_j^n must be indifferent between all the points of this interval:

$$P(t_i^g)\, U_{j,n}^L + P(t_i^n)\left(t_j^n G_i\left(t_i^n, e_j^n\right) + U_{j,n}^L\right) - e_j^n =$$
$$= P(t_i^g)\, U_{j,n}^L + P(t_i^n)\left(t_j^n G_i\left(t_i^n, \tilde{e}_j^n\right) + U_{j,n}^L\right) - \tilde{e}_j^n$$

When substitute for $G_i(t_i^n, e)$, the simplified expression becomes:

$$\left(e_j^n - \tilde{e}_j^n\right)\left[\frac{t_j^n}{t_j^g} - 1\right] = 0$$

Since $\left[\frac{t_j^n}{t_j^g}-1\right]<0$, this equality holds iff $e_j^n=\tilde{e}_j^n$, a contradiction. Hence, t_j^n must place the atom of size 1 at $e\geq 0$.

Now assume t_j^n always plays $e_j^n=q>0$, $q\in(0,\,D_1)$ in equilibrium, and this results in expected payoff $\pi_{j,\,n}(q)$:

$$\pi_{j,\,n}(q)=P(t_i^n)\,U_{j,n}^L+P(t_i^n)\left(t_j^nG_i(t_i^n,\,q)+U_{j,n}^L\right)-q$$

Given that q is a best reply, it must be $\pi_{j,\,n}(q)\geq\pi_{j,\,n}(0)=U_{j,n}^L$ where $\pi_{j,\,n}(0)$ is a payoff when t_j^n chooses $e_j^n=0$:

$$\pi_{j,\,n}(q)\geq\pi_{j,\,n}(0)\Leftrightarrow q\left[\frac{t_j^n}{t_j^g}-1\right]\geq 0$$

As $\left[\frac{t_j^n}{t_j^g}-1\right]<0$ always holds, the condition is never satisfied for $q>0$: $e_j^n=0$ dominates $e_j^n=q$ for any $q>0$. Hence, t_j^n must place $p_3^0\left(t_j^n\right)=1$ at $e=0$ in equilibrium.

□

Overall, for $P\left(t_j^g\right)\geq\tilde{T}$ the equilibrium partition and contestants' strategies look become:

- $(D_1,\,D_2]$: t_i^g and t_j^g randomize uniformly on the interval with probabilities 1 and $\left\{\frac{t_j^g}{t_i^g}\right\}$, respectively;
- $(0,\,D_1]$: t_i^n and t_j^g randomize uniformly on the interval with probabilities 1 and $\left\{\frac{P\left(t_i^n\right)t_j^g}{P\left(t_j^g\right)t_i^n}\right\}$, respectively;
- t_j^g and t_j^n place $p_1^0\left(t_j^g\right)\in(0,\,1)$ and $p_3^0\left(t_j^n\right)=1$ at $e=0$, respectively.

2. $t_i^n\leq 0$.

First, $t_i^n\leq 0$ will always place $p_3^0\left(t_i^n\right)=1$ at $e=0$ in equilibrium (the argument is similar to ***Lemma 1***). Then t_j^g competing against $t_i^n\leq 0$ must choose $p_2^0\left(t_j^g\right)=1-\frac{t_j^g}{t_i^g}\in(0,\,1)$ to exhaust his bidding probability (the argument is similar to ***Lemma 3***). Finally, t_j^n plays $e=0$ with probability 1 (***Lemma 4***). As a result, the equilibrium partition consists of one interval $[0,\,L_{top}]$:

- t_i^g and t_j^g randomize uniformly on $(0,\,L_{top}]$ with probabilities 1 and $\left\{\frac{t_j^g}{t_i^g}\right\}$, respectively;
- t_j^g places $p_2^0\left(t_j^g\right)\in(0,\,1)$ at $e=0$; t_i^n and t_j^n always choose $e=0$.

To show the uniqueness, I refer to Siegel (2014). He proved that the "top-down" algorithm delivers the unique equilibrium of the specified game if $P\left(t_j|\,t_i\right)t_i$ increases in t_i for any t_j, $i,\,j=\{1,\,2\}$:

$$P\left(t_j^g \middle|\, t_i^g\right) t_i^g > P\left(t_j^g \middle|\, t_i^n\right) t_i^n,\ P\left(t_j^n \middle|\, t_i^g\right) t_i^g > P\left(t_j^n \middle|\, t_i^n\right) t_i^n,\ i \neq j$$

Since in this case realizations of α_1 and α_2 are independent, it must be $P(t_j|\, t_i) = P(t_j)$. Then the condition reduces to $t_i^g > t_i^n$ that holds by definition. Thus, the partitions characterized above correspond to a unique equilibrium of the game for given preferences, types' probability distribution, and prize schedules.

$\square$

Proposition 1.5. *For $\beta_1 > \beta_2$ and $\underline{\alpha}_1 > \bar{\alpha}_2 + \beta_1 - \beta_2$ there exists a non-empty subset of R_k^s, $R_k^{L,s}$, such that for any probability–valuation profile in $R_k^{L,s}$ the designer uses both goods completely and assigns a positive losing prize in dimension A.*

Proof. The problem of the designer was introduced in Proposition 1.2. Define $J\left(A^W, A^L, B^W, B^L\right) \equiv J\left(\triangle_A, \triangle_B\right)$. Depending on the probability-valuation profile and the prize scheme, the equilibrium partition in the game between contestants looks differently (see Proposition 1.4). Moreover, expected aggregate effort exhibits high-order non-linearity in prize spreads ($\triangle_A$ and $\triangle_B$), and R_k^L cannot be characterized completely. To get analytical results, I look only at asymmetric preference profiles ($\bar{\alpha}_1 > \bar{\alpha}_2$, $\bar{\alpha}_1 < \beta_2$), assume $\underline{\alpha}_1 > \underline{\alpha}_2$ and characterize $\bar{R}_k^L$, a non-empty subset of R_k^L.

Suppose $\triangle_B = 1$ is optimal (verify this later). Also assume no switch in contestants' identities for any feasible $\triangle_A$:

$$t_1^j\left(\triangle_A, 1\right) > t_2^j\left(\triangle_A, 1\right) \,\forall\, \triangle_A \in [-1, 1]\,,\, j = \{g, n\} \Leftrightarrow$$
$$\Leftrightarrow \max\left\{\tilde{\triangle}_A^g(1)\,,\, \tilde{\triangle}_A^n(1)\right\} < -1 \Leftrightarrow \begin{cases} \bar{\alpha}_1 < \bar{\alpha}_2 + \beta_1 - \beta_2 \\ \underline{\alpha}_1 < \underline{\alpha}_2 + \beta_1 - \beta_2 \end{cases}$$
$$\tilde{\triangle}_A^n(1) = \tfrac{\bar{\alpha}_2 - \bar{\alpha}_1}{\beta_1 - \beta_2},\ \tilde{\triangle}_A^g(1) = \tfrac{\underline{\alpha}_1 - \underline{\alpha}_2}{\beta_1 - \beta_2}$$

To make the analytical solution tractable, I fix the following parametrization:

$$\bar{\alpha}_1 = 0.5,\ \bar{\alpha}_2 = 0.1,\ \beta_1 = 2,\ \beta_2 = 1$$

and leave $\underline{\alpha}_1$, $\underline{\alpha}_2$ and k free.

Since α_i has two possible realizations, $P\left(t_i^g\right)$ must depend on $\triangle_A$:

$$\triangle_A \in [0, 1] \Rightarrow t_i^g = \bar{\alpha}_i \triangle_A + \beta_i,\ P\left(t_i^g\right) = k$$
$$\triangle_A \in [-1, 0) \Rightarrow t_i^g = \underline{\alpha}_i \triangle_A + \beta_i,\ P\left(t_i^g\right) = 1 - k$$

and $J(\cdot)$ changes when passes $\triangle_A = 0$.

There are two equilibrium configurations that support $t_1^j\left(\triangle_A, 1\right) > t_2^j\left(\triangle_A, 1\right)$, $j = \{g, n\}$ (all notations are specified in the proof of Proposition 1.4):

1. E_1 with $P\left(t_2^g\right) \in \left[0, \tilde{T}\left(\triangle_A, \triangle_B\right)\right)$:
   - t_1^n randomizes uniformly on $[0, T_1]$ and $(T_1, T_2]$ with probabilities $p\left(t_1^n\right)$ and $1 - p\left(t_1^n\right)$, respectively, $p\left(t_1^n\right) = \frac{P\left(t_2^g\right)\left(t_1^g - t_2^g\right)t_1^n}{t_1^g t_2^g P\left(t_1^n\right)}$;
   - t_2^n uses a strategy including uniform randomization on $(0, T_1]$ and the atom $p_1^0\left(t_2^n\right) \in (0, 1)$ at $e = 0$;

- t_1^g randomizes uniformly on $(T_2,\, T_3]$ with probability 1;
- t_2^g randomizes uniformly on $(T_1,\, T_2]$ and $(T_2,\, T_3]$ with probabilities $p\left(t_2^g\right)$ and $1 - p\left(t_2^g\right)$, respectively, $p\left(t_2^g\right) = \frac{t_2^g}{t_1^g}$.

2. E_2 with $P\left(t_2^g\right) \in \left[\tilde{T}\left(\triangle_A,\, \triangle_B\right),\, 1\right]$:

- t_1^n randomizes uniformly on $[0,\, D_1]$ with probability 1;
- t_2^n places the atom $p_3^0\left(t_j^n\right) = 1$ at $e = 0$;
- t_1^g randomizes uniformly on $(D_1,\, D_2]$ with probability 1;
- t_2^g randomizes uniformly on $[0,\, D_1]$ and $(D_1,\, D_2]$ with probabilities $p_1\left(t_2^g\right)$ and $p_2\left(t_2^g\right)$ respectively, and places the atom $\left\{1 - p_1\left(t_2^g\right) - p_2\left(t_2^g\right)\right\}$ at $e = 0$, $p_1\left(t_2^g\right) = \frac{P\left(t_1^n\right)t_2^g}{P\left(t_2^g\right)t_1^n}$, $p_2\left(t_2^g\right) = \frac{t_2^g}{t_1^g}$.

First, take $\triangle_A \in [0,\, 1]$, i.e. $P\left(t_i^g\right) = k$. Also suppose $k \in \left[0,\, \tilde{T}\left(\triangle_A,\, 1\right)\right)$ for any $\triangle_A \in [0,\, 1]$, and the equilibrium configuration E_1 realizes. The "Winner–Takes–All" schedule is not optimal if $\frac{\partial J}{\partial \triangle_A}(1,\, 1) < 0$:

$$\frac{\partial J}{\partial \triangle_A}(1,\, 1) < 0 \Leftrightarrow \gamma_1 k^2 + \gamma_2 k + \gamma_3 < 0$$

where γ_i, $i = \{1,\, 2,\, 3\}$ are functions of $\underline{\alpha}_1$ and $\underline{\alpha}_2$ that can take both negative and positive values.

Lemma 1. *There exist positive $\tilde{\underline{\alpha}}_2\left(\underline{\alpha}_1\right)$ (the function of $\underline{\alpha}_1$) and $\tilde{k}$ such that $\frac{\partial J}{\partial \triangle_A}(1,\, 1) < 0$ holds for any $\underline{\alpha}_2 \in \left[0,\, \tilde{\underline{\alpha}}_2\left(\underline{\alpha}_1\right)\right)$ and $k \in \left[0,\, \tilde{k}\right)$.*

Proof. Consider γ_i, $i = \{1,\, 2,\, 3\}$ as the functions of $\underline{\alpha}_2$. Then γ_1 looks as follows:

$$\gamma_1 = \delta_1^\gamma \underline{\alpha}_2^2 + \delta_2^\gamma \underline{\alpha}_2 + \delta_3^\gamma$$
$$\delta_1^\gamma = \textstyle\sum_{j=0}^3 c_{1j}\underline{\alpha}_1^j,\ \delta_l^\gamma = \sum_{j=0}^4 c_{lj}\underline{\alpha}_1^j,\ l = \{2,\, 3\}$$
$$c_{1j},\, c_{3j} > 0,\ c_{2j} < 0\, \forall\, j$$

where $\gamma_1 = 0$ always has two positive real roots, $u_{\gamma_1}^1$ and $u_{\gamma_1}^2$, $u_{\gamma_1}^1 < u_{\gamma_1}^2$ such that $u_{\gamma_1}^2 > \underline{\alpha}_1$ and $u_{\gamma_1}^1 > \underline{\alpha}_1$ for any $\underline{\alpha}_1 \in [0,\, \bar{\alpha}_1)$ and $\underline{\alpha}_1 \in (0.12,\, \bar{\alpha}_1)$, respectively. Hence, under $\underline{\alpha}_1 \in (0.12,\, \bar{\alpha}_1)$ the condition $\underline{\alpha}_1 > \underline{\alpha}_2$ implies $\gamma_1 > 0$.

Analyzing γ_2 and γ_3 delivers:

$$\gamma_2 = \mu_1^\gamma \underline{\alpha}_2^2 + \mu_2^\gamma \underline{\alpha}_2 + \mu_3^\gamma \qquad \gamma_3 = \varphi_1^\gamma \underline{\alpha}_2^2 + \varphi_2^\gamma \underline{\alpha}_2 + \varphi_3^\gamma$$
$$\mu_l^\gamma = \textstyle\sum_{j=0}^2 d_{lj}\underline{\alpha}_1^j,\ l = \{1,\, 2,\, 3\} \qquad \varphi_1^\gamma = \sum_{j=0}^1 s_{1j}\underline{\alpha}_1^j,\ \varphi_2^\gamma = \sum_{j=0}^2 s_{2j}\underline{\alpha}_1^j,\ \varphi_3^\gamma = s_3\underline{\alpha}_1^j$$
$$d_{1j},\, d_{2j} < 0,\ d_{3j} > 0\, \forall\, j \qquad s_{1j},\, s_{2j} > 0,\ s_3 < 0\, \forall\, j$$

where $\gamma_2 > 0$ ($\gamma_3 > 0$) holds under $\underline{\alpha}_2 \in [0,\, u_{\gamma_2})$ ($\underline{\alpha}_2 \in [u_{\gamma_3},\, \min\{\underline{\alpha}_1,\, \bar{\alpha}_2\})$) and u_{γ_2} (u_{γ_3}) corresponds to the positive real root of $\gamma_2 = 0$ ($\gamma_3 = 0$). Finally, $u_{\gamma_2} > \underline{\alpha}_1$ is satisfied with $\underline{\alpha}_1 \in [0,\, 0.003)$ and $u_{\gamma_3} < \underline{\alpha}_1$ for any feasible $\underline{\alpha}_1$.

Further I investigate how u_{γ_2} and u_{γ_3} relate to each other. Both functions monotonically increase in $\underline{\alpha}_1$, but u_{γ_2} changes faster.[89] As a result, it must be $u_{\gamma_2} > u_{\gamma_3}$, and for $\underline{\alpha}_1 \in (0.12,\, \bar{\alpha}_1)$ one can sign γ_i, $i = \{1,\, 2,\, 3\}$ as follows:

$$\begin{gathered}
\underline{\alpha}_2 \in [0,\, u_{\gamma_3}) \Rightarrow \gamma_{1,\,2} > 0,\, \gamma_3 < 0 \\
\underline{\alpha}_2 \in [u_{\gamma_3},\, u_{\gamma_2}) \Rightarrow \gamma_l > 0,\, l = \{1,\, 2,\, 3\} \\
\underline{\alpha}_2 \in [u_{\gamma_2},\, \min\{\underline{\alpha}_1,\, \bar{\alpha}_2\}) \Rightarrow \gamma_{1,\,3} > 0,\, \gamma_2 < 0
\end{gathered}$$

Define $\tilde{\underline{\alpha}}_2(\underline{\alpha}_1) = u_{\gamma_3}$ and take $\underline{\alpha}_2 \in [0,\, \tilde{\underline{\alpha}}_2(\underline{\alpha}_1))$. Then equation $\gamma_1 k^2 + \gamma_2 k + \gamma_3 = 0$ has one positive real root $\hat{k}$ such that $\gamma_1 k^2 + \gamma_2 k + \gamma_3 < 0$ for $k \in \left[0,\, \hat{k}\right)$. Since the equilibrium structure E_1 requires $k \in \left[0,\, \tilde{T}(1,\, 1)\right)$, imposing $\tilde{k} = \min\left\{\hat{k},\, \tilde{T}(1,\, 1)\right\}$ gives the lemma.

□

Using the parameter sets specified in ***Lemma 1***, I fix $\underline{\alpha}_1 = 0.15$ and $\underline{\alpha}_2 = 0.01$ but leave k free. Since $\tilde{T}(1,\, 1)$ exceeds $\hat{k}$ under imposed restrictions, it must be $\tilde{k} = \hat{k} = 0.12$.

Lemma 2. *$\tilde{T}(\triangle_A,\, 1)$ decreases in $\triangle_A$ for any $\triangle_A \in [0,\, 1]$.*

Proof. $\frac{\partial \tilde{T}(\triangle_A,\, 1)}{\partial \triangle_A} > 0$ holds iff $\triangle_A \in (-\infty,\, -6.6) \cup (13.8)$ that has an empty intersection with $\triangle_A \in [0,\, 1]$.[90] Then $\frac{\partial \tilde{T}(\triangle_A,\, 1)}{\partial \triangle_A} < 0$ for any $\triangle_A \in [0,\, 1]$ follows. □

Lemma 2 implies that $\tilde{T}(\triangle_A,\, 1)$ has its maximum at $\triangle_A = 0$, and $\tilde{T}(0,\, 1) = \frac{1}{2}$. As a result, $k \in \left[0,\, \tilde{T}(1,\, 1)\right)$ is the strictest target, and no additional constraint on k is needed when $\triangle_A \in [0,\, 1]$.

Further, I check how $\frac{\partial J}{\partial \triangle_A}(\triangle_A,\, 1)$ behaves with respect to $\triangle_A \in [0,\, 1]$.

Lemma 3. *$\frac{\partial J}{\partial \triangle_A}(\triangle_A,\, 1)$ increases in $\triangle_A \in [0,\, 1]$.*

Proof. $\frac{\partial^2 J}{\partial \triangle_A^2}(\triangle_A,\, 1)$ is non-negative iff:

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[89] Derivatives of u_{γ_2} and u_{γ_3} with respect to $\underline{\alpha}_1$ look as follows:

$$\begin{gathered}
\frac{\partial u_{\gamma_2}}{\partial \underline{\alpha}_1} > 0 \,\forall\, \underline{\alpha}_1 \in [0,\, 4.79) \\
\frac{\partial u_{\gamma_3}}{\partial \underline{\alpha}_1} > 0 \,\forall\, \underline{\alpha}_1 \in [0,\, 2.9)
\end{gathered}$$

that implies $\frac{\partial u_{\gamma_l}}{\partial \underline{\alpha}_1} > 0$, $l = \{2,\, 3\}$ for any feasible $\underline{\alpha}_1$ (remember $\underline{\alpha}_1 < \bar{\alpha}_1 = 0.5$). Also $\frac{\partial u_{\gamma_2}}{\partial \underline{\alpha}_1} > \frac{\partial u_{\gamma_3}}{\partial \underline{\alpha}_1}$ holds under $\underline{\alpha}_1 \in [0,\, 7.84)$, and it must be $\frac{\partial u_{\gamma_2}}{\partial \underline{\alpha}_1} > \frac{\partial u_{\gamma_3}}{\partial \underline{\alpha}_1}$ for any feasible $\underline{\alpha}_1$ follows.

[90] A complete derivative is $\frac{\partial \tilde{T}(\triangle_A,\, 1)}{\partial \triangle_A} = \frac{-5\sum_{j=0}^{2} w_j \triangle_A^j}{[z(\triangle_A)]^2}$ where $w_1 < 0$, $w_{2,\,3} > 0$ and $z(\triangle_A)$ is a function of $\triangle_A$.

$$\pi_1 k^2 + \pi_2 k + \pi_3 \geq 0$$
$$\pi_l = \sum_{j=0}^{8} q_{lj} \triangle_A^j,\ l = \{1,\, 2,\, 3\},\ q_{1j},\, q_{3j} > 0\,\forall\, j$$
$$\pi_2 > 0\,\forall\, \triangle_A \in [0,\, 1]$$

and the inequality holds for any $k \geq 0$.[91] As a result, $\frac{\partial J}{\partial \triangle_A}(\triangle_A,\, 1)$ must increase in $\triangle_A \in [0,\, 1]$. □

Next, consider $\triangle_A \in [-1,\, 0)$. Since $P(t_i^g)$ and contestants' types switch at $\triangle_A = 0$, the equilibrium configuration in the interval of interest can differ. Suppose E_1 is played in $\triangle_A \in [-1,\, 0)$. Then $P(t_i^g) = 1 - k$ must belong to $\left[0,\, \tilde{T}_s(\triangle_A,\, \triangle_B)\right)$ where $\tilde{T}_s(\cdot)$ reflects the probability threshold in the case of changed types. Rewritten in terms of k, the condition becomes $k \in \left(1 - \tilde{T}_s(\cdot),\, 1\right]$.

Lemma 4. *$\tilde{T}_s(\triangle_A,\, 1)$ decreases in $\triangle_A$ for any $\triangle_A \in [-1,\, 0)$.*

Proof. $\frac{\partial \tilde{T}_s(\triangle_A,\, 1)}{\partial \triangle_A} > 0$ holds iff $\triangle_A \in (-\infty,\, -19.8) \cup (-5.7)$ that has a non-empty intersection with $\triangle_A \in (-1,\, 0]$.[92] Then $\frac{\partial \tilde{T}_s(\triangle_A,\, 1)}{\partial \triangle_A} < 0$ for any $\triangle_A \in [-1,\, 0)$ follows. □

Given ***Lemma 4***, $\left\{1 - \tilde{T}_s(\cdot)\right\}$ must increase in $\triangle_A \in [-1,\, 0)$:

$$\sup\left\{1 - \tilde{T}_s(\triangle_A,\, 1)\right\} = 1 - \tilde{T}_s(0,\, 1) = \tfrac{1}{2}$$
$$\inf\left\{1 - \tilde{T}_s(\triangle_A,\, 1)\right\} = 1 - \tilde{T}_s(-1,\, 1) = 0.41$$

However, the sets $k \in \left(1 - \tilde{T}_s(0,1),\, 1\right]$ and $k \in \left[0,\, \tilde{k}\right)$ have an empty intersection. Hence, $\frac{\partial J}{\partial \triangle_A}(1,\, 1) < 0$ and E_1- type equilibrium in $\triangle_A \in [-1,\, 0)$ result in a contradiction.

Now assume E_2 is played in $\triangle_A \in [-1,\, 0)$. Then the relevant equilibrium constrain on k becomes $k \in \left[0,\, 1 - \tilde{T}_s(\cdot)\right]$, and this is implied by $k \in \left[0,\, \tilde{k}\right)$:

$$\inf\left\{1 - \tilde{T}_s(\triangle_A,\, 1)\right\} > \tilde{k}$$

Let $\frac{\partial J^-}{\partial \triangle_A}(\triangle_A,\, 1)$ be the derivative of $J(\cdot)$ with respect to $\triangle_A \in [-1,\, 0)$.

Lemma 5. *$\frac{\partial J^-}{\partial \triangle_A}(\triangle_A,\, 1)$ increases in $\triangle_A \in [-1,\, 0)$.*

Proof. $\frac{\partial^2 J^-}{\partial \triangle_A^2}(\triangle_A,\, 1)$ looks as follows:

$$\frac{\partial^2 J^-}{\partial \triangle_A^2}(\triangle_A,\, 1) = \frac{\sum_{j=0}^{3} y_{1j} \triangle_A^j k^2 + \sum_{j=0}^{3} y_{2j} \triangle_A^j}{\left(\sum_{j=0}^{2} y_{3j} \triangle_A^j\right)^3}$$
$$y_{2j},\, y_{3j} > 0\,\forall\, j,\ y_{11} < 0$$

[91] When $\triangle_A \geq 0$, positive entries of π_2 over-compensate negative ones.

[92] A complete derivative is $\frac{\partial \tilde{T}_s(\triangle_A,\, 1)}{\partial \triangle_A} = \frac{-40 \sum_{j=0}^{2} w_{sj} \triangle_A^j}{[z_s(\triangle_A)]^2}$ where $w_{sj} > 0\ \forall\ j$ and $z_s(\triangle_A)$ is a function of $\triangle_A$.

where polynomial coefficients are positive under $\triangle_A \in [-1,\, 0)$.[93] Hence, $\frac{\partial J^-}{\partial \triangle_A}(\triangle_A,\, 1)$ must increase in $\triangle_A \in [-1,\, 0)$ for any feasible k.

$\square$

Since $\frac{\partial J^-}{\partial \triangle_A}(\triangle_A,\, 1)$ is continuous (no switching points were assumed) and increasing in $\triangle_A \in [-1,\, 0)$, the condition $\frac{\partial J^-}{\partial \triangle_A}(0,\, 1) < 0$ is sufficient for $J(\triangle_A,\, 1)$ to decrease in $\triangle_A \in [-1,\, 1]$:[94]

$$\frac{\partial J^-}{\partial \triangle_A}(0,\, 1) = -m_1 k^2 - m_2,\; m_i > 0,\; i = \{1,\, 2\}$$

As a result, for $k \in \left[0,\, \tilde{k}\right)$ the designer's objective $J(\triangle_A,\, 1)$ decreases in $\triangle_A \in [-1,\, 1]$. Thus, it is optimal to choose $\triangle_A = -1$ given $\triangle_B = 1$ and use good A's endowment completely.

Further I verify the optimality of $\triangle_B = 1$. The proposed valuation structure features feasible identity switching and cutoff points:

1. $\triangle_A \geq 0$: type t_2^n is well-defined only for $\triangle_B \in \left[-\frac{\alpha_2}{\beta_2}\triangle_A,\, 1\right]$, $\left\{-\frac{\alpha_2}{\beta_2}\triangle_A\right\} \in (-1,\, 0)$, and no switch in contestants' power takes place:
$$\tilde{\triangle}_B^g(\triangle_A) < \tilde{\triangle}_B^n(\triangle_A) = \frac{\alpha_2 - \alpha_1}{\beta_1 - \beta_2}\triangle_A < -\frac{\alpha_2}{\beta_2}\triangle_A \,\forall\, \triangle_A \geq 0$$

2. $\triangle_A < 0$: type t_1^n is well-defined only for $\triangle_B \in \left[-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A,\, 1\right]$, $\left\{-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A\right\} \in (0,\, 1)$, and the power of n types switches on the interval:
$$\tilde{\triangle}_B^g(\triangle_A) < -\frac{\bar{\alpha}_1}{\beta_1}\triangle_A < \tilde{\triangle}_B^n(\triangle_A) = \frac{\bar{\alpha}_2 - \bar{\alpha}_1}{\beta_1 - \beta_2}\triangle_A \,\forall\, \triangle_A < 0$$
and for $\triangle_B \in \left[-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A,\, \tilde{\triangle}_B^n(\triangle_A)\right)$ type n of contestant 2 has stronger incentives to win.

Since β_i is deterministic, $P(t_i^n)$ does not change when $J(\triangle_A,\, \triangle_B)$ passes $\triangle_B = 0$ for given $\triangle_A$. First, take the case of $\triangle_A \geq 0$ when E_1 is played. Let $\frac{\partial J}{\partial \triangle_B}\left(\triangle_A^+,\, \triangle_B\right)$ be the derivative of $J(\cdot)$ with respect to $\triangle_B$ when $\triangle_A \geq 0$.

Lemma 6. *There exists $\bar{k} \in (0,\, 1]$ such that for any $k \in \left[0,\, \bar{k}\right]$ the derivative $\frac{\partial J}{\partial \triangle_B}\left(\triangle_A^+,\, \triangle_B\right)$ increases in $\triangle_B$, $\triangle_B \in \left[-\frac{\alpha_2}{\beta_2}\triangle_A^+,\, 1,\, 1\right]$.*

Proof. $\frac{\partial^2 J}{\partial \triangle_B^2}\left(\triangle_A^+,\, \triangle_B\right)$ looks as follows:

$$\frac{\partial^2 J}{\partial \triangle_B^2}\left(\triangle_A^+,\, \triangle_B\right) = \frac{\nu_1 k^2 + \nu_2 k + \nu_3}{(3\triangle_A + 40\triangle_B)^3 [z(\triangle_A, \triangle_B)]^4}$$
$$\nu_i = \sum_{j=0}^{8} h_{ij}\triangle_A^j \triangle_B^{8-j},\; i = \{1,\, 2,\, 3\}$$
$$h_{1j},\, h_{3j} > 0 \,\forall\, j$$

[93] $\sum_{j=0}^{3} y_{1j}\triangle_A^j$ has a single real root $\triangle_A^1 = 19.6$ such that $\sum_{j=0}^{3} y_{1j}\triangle_A^j > 0$ for any $\triangle_A < \triangle_A^1$. Coefficient $\sum_{j=0}^{3} y_{2j}\triangle_A^j$ reduces to $(\triangle_A + 4)^3$ and is always positive under $\triangle_A \in [-1,\, 0)$. Polynomial $\sum_{j=0}^{2} y_{3j}\triangle_A^j$ has two negative real roots located below $\triangle_A = -1$.

[94] Continuity of $\frac{\partial J^-}{\partial \triangle_A}(\triangle_A,\, 1)$ and $\frac{\partial J^-}{\partial \triangle_A}(0,\, 1) < 0$ imply $\frac{\partial J^-}{\partial \triangle_A}(\triangle_A,\, 1) < 0$ in a neighborhood to the left of $\triangle_A = 0$.

Since $\triangle_B$ cannot be lower than $\left\{-\frac{\alpha_2}{\beta_2}\triangle_A^+,\, 1\right\}$, the denominator of this expression is always positive. Moreover, both $\nu_1 > 0$ and $\nu_3 > 0$ hold for any feasible $\triangle_B$,[95] but the sign of ν_2 is ambiguous. If $\nu_2 > 0$ or $\nu_2 < 0$, but the discriminant of $\nu_1 k^2 + \nu_2 k + \nu_3 = 0$ is negative, any $k \in [0,\, 1]$ results in $\frac{\partial^2 J}{\partial \triangle_B^2}\left(\triangle_A^+,\, \triangle_B\right) > 0$, and one can take $\bar{k} = 1$. Otherwise, $\nu_2 < 0$ combined with the positive discriminant delivers two real roots of $\nu_1 k^2 + \nu_2 k + \nu_3 = 0$, k_1 and k_2, $k_1 < k_2$, such that $\frac{\partial^2 J}{\partial \triangle_B^2}\left(\triangle_A^+,\, \triangle_B\right) \geq 0$ holds for $k \in [0,\, k_1] \cup [k_2,\, \infty)$. Then taking $\bar{k} = \min\{1,\, k_1\}$ gives the lemma.

□

Using ***Lemma 6***, I restrict $k \in \left[0,\, \min\left\{\tilde{k},\, \bar{k}\right\}\right)$. Given that $\frac{\partial J}{\partial \triangle_B}\left(\triangle_A^+,\, \triangle_B\right)$ is continuous and increasing in $\triangle_B \in \left[-\frac{\alpha_2}{\beta_2}\triangle_A^+,\, 1\right]$, the condition $\frac{\partial J}{\partial \triangle_B}\left(\triangle_A^+,\, -\frac{\alpha_2}{\beta_2}\triangle_A^+\right) > 0$ is sufficient to guarantee $\frac{\partial J}{\partial \triangle_B}\left(\triangle_A^+,\, \triangle_B\right) > 0$ for any $\triangle_A^+$ and feasible $\triangle_B$:

$$\frac{\partial J}{\partial \triangle_B}\left(\triangle_A^+,\, \triangle_B\right) > 0 \Leftrightarrow \tau_1 k^2 + \tau_2 k + \tau_3 > 0$$

where $\tau_1 < 0$, $\tau_{2,3} > 0$. The inequality holds for $k \in (-0.4,\, 2.5)$, and this set includes all feasible values of k. Hence, $\frac{\partial J}{\partial \triangle_B}\left(\triangle_A^+,\, \triangle_B\right)$ is positive for any $\triangle_A^+$, $\triangle_B \in \left[-\frac{\alpha_2}{\beta_2}\triangle_A^+,\, 1\right]$ and $k \in \left[0,\, \min\left\{\tilde{k},\, \bar{k}\right\}\right)$.

Next, I analyze the case of $\triangle_A < 0$ where E_2 is played. Define $\frac{\partial J}{\partial \triangle_B}\left(\triangle_A^-,\, \triangle_B\right)$ as the derivative of $J(\cdot)$ with respect to $\triangle_B$. Since type n of contestant 2 has stronger incentives to win when $\triangle_B \in \left[-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-,\, \tilde{\triangle}_B^n\left(\triangle_A^-\right)\right)$, one must consider this interval and $\triangle_B \in \left[\tilde{\triangle}_B^n\left(\triangle_A^-\right),\, 1\right)$ separately. Let E_2^+ and E_2^- ($\frac{\partial J^+}{\partial \triangle_B}\left(\triangle_A^-,\, \triangle_B\right)$ and $\frac{\partial J^-}{\partial \triangle_B}\left(\triangle_A^-,\, \triangle_B\right)$) correspond to the equilibria (the derivatives) to the right and to the left of $\tilde{\triangle}_B^n\left(\triangle_A^-\right)$, respectively. Define $\tilde{T}_s^r\left(\triangle_A^-,\, \triangle_B\right)$ as the probability threshold for $\triangle_A^-$ and $\triangle_B \in \left[-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-,\, \tilde{\triangle}_B^n\left(\triangle_A^-\right)\right)$.

Lemma 7. *$\tilde{T}_s\left(\triangle_A^-,\, \triangle_B\right)$ and $\tilde{T}_s^r\left(\triangle_A^-,\, \triangle_B\right)$ decrease in $\triangle_B$ for any $\triangle_B \in \left[\tilde{\triangle}_B^n\left(\triangle_A^-\right),\, 1\right]$ and $\triangle_B \in \left[-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-,\, \tilde{\triangle}_B^n\left(\triangle_A^-\right)\right)$, respectively.*

Proof. $\frac{\partial \tilde{T}_s\left(\triangle_A^-,\, \triangle_B\right)}{\partial \triangle_B}$ and $\frac{\partial \tilde{T}_s^r\left(\triangle_A^-,\, \triangle_B\right)}{\partial \triangle_B}$ look as follows:

$$\frac{\partial \tilde{T}_s\left(\triangle_A^-,\, \triangle_B\right)}{\partial \triangle_B} = \frac{\triangle_A^- \varphi_1\left(\triangle_A^-,\, \triangle_B\right)}{\left[x_1\left(\triangle_A^-,\, \triangle_B\right)\right]^2}$$

$$\frac{\partial \tilde{T}_s^r\left(\triangle_A^-,\, \triangle_B\right)}{\partial \triangle_B} = \frac{\triangle_A^- \varphi_2\left(\triangle_A^-,\, \triangle_B\right)}{\left[x_2\left(\triangle_A^-,\, \triangle_B\right)\right]^2}$$

$$\varphi_l\left(\triangle_A^-,\, \triangle_B\right) = \sum_{j=0}^{2} y_{lj} \triangle_A^j \triangle_B^{2-j},\ l = \{1,\, 2\}$$

where $y_{lj} > 0$ for any l, j and $x_l\left(\triangle_A^-,\, \triangle_B\right)$ is the function of $\triangle_A^-$, $\triangle_B$. Solving $\varphi_1\left(\triangle_A^-,\, \triangle_B\right)$ with

[95] When $\triangle_B \geq 0$, all elements of the polynomials exceed zero. When $\triangle_B < 0$, positive entries over-compensate negative ones corresponding to odd powers.

respect to $\triangle_B$ brings that $\varphi_s\left(\triangle_A^-,\, \triangle_B\right)$ is positive for any $\triangle_B > -0.17\triangle_A$, and this includes all feasible values of $\triangle_B$. Similarly, $\varphi_1\left(\triangle_A^-,\, \triangle_B\right) > 0$ holds under $\triangle_B > -0.11\triangle_A$ that covers $\triangle_B \in \left[-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-,\, \tilde{\triangle}_B^n\left(\triangle_A^-\right)\right)$. Thus, for $\triangle_A^- \in [-1,\, 0)$ it must be $\frac{\partial \tilde{T}_s\left(\triangle_A^-,\triangle_B\right)}{\partial \triangle_B} < 0$ and $\frac{\partial \tilde{T}_s^r\left(\triangle_A^-,\triangle_B\right)}{\partial \triangle_B} < 0$ for any $\triangle_B \in \left[\tilde{\triangle}_B^n\left(\triangle_A^-\right),\, 1\right]$ and $\triangle_B \in \left[-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-,\, \tilde{\triangle}_B^n\left(\triangle_A^-\right)\right)$, respectively.

□

Lemma 7 implies that $\left\{1 - \tilde{T}_s\left(\triangle_A^-,\, \triangle_B\right)\right\}$ and $\left\{1 - \tilde{T}_s^r\left(\triangle_A^-,\, \triangle_B\right)\right\}$ must increase in $\triangle_B$ and have infima at $\triangle_B = \tilde{\triangle}_B^n\left(\triangle_A^-\right)$ and $\triangle_B = -\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-$, respectively:

$$\inf\left\{1 - \tilde{T}_s\left(\triangle_A^-,\, \triangle_B\right)\right\} = 0.24$$
$$\inf\left\{1 - \tilde{T}_s^r\left(\triangle_A^-,\, \triangle_B\right)\right\} = 0.16$$

Given that $\tilde{k}$ is smaller than any of these values, $k \in \left[0,\, \min\left\{\tilde{k},\, \bar{k}\right\}\right)$ implies both $k \in \left[0,\, 1 - \tilde{T}_s\left(\triangle_A^-,\, \triangle_B\right)\right]$ and $k \in \left[0,\, 1 - \tilde{T}_s^r\left(\triangle_A^-,\, \triangle_B\right)\right]$. Hence, no additional restrictions are needed on k.

Lemma 8. *$\frac{\partial J^+}{\partial \triangle_B}\left(\triangle_A^-,\, \triangle_B\right)$ and $\frac{\partial J^-}{\partial \triangle_B}\left(\triangle_A^-,\, \triangle_B\right)$ increase in $\triangle_B$ for any feasible $\triangle_B$ and $\triangle_A^-$.*

Proof. Consider $\frac{\partial^2 J^-}{\partial \triangle_B^2}\left(\triangle_A^-,\, \triangle_B\right)$:

$$\frac{\partial^2 J^-}{\partial \triangle_B^2}\left(\triangle_A^-,\, \triangle_B\right) = \frac{\rho_1 k^2 + (\triangle_A + 10\triangle_B)^3}{(3\triangle_A + 40\triangle_B)^3 (\triangle_A + 10\triangle_B)^3}$$
$$\rho_1 = \sum_{j=0}^{3} x_{1j}\triangle_A^j \triangle_B^{3-j},\; x_{1j} > 0\,\forall\, j$$

The denominator and the second component of the numerator increase in $\triangle_B$, are positive at $\triangle_B = \triangle_B^{min}$, $\triangle_B^{min} = \left\{-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-\right\}$ and, consequently, any feasible $\triangle_B$. Consider ρ_1 as the function of $\triangle_B$ (recall $\triangle_A^- < 0$):

$$\frac{\partial \rho_1\left(\triangle_B\right)}{\partial \triangle_B} \leq 0 \Leftrightarrow \triangle_B \in [-0.05,\, -0.08]\,\triangle_A^-$$

The set has an empty intersection with $\triangle_B \in \left[-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-,\, \tilde{\triangle}_B^n\left(\triangle_A^-\right)\right)$, and ρ_1 must increase in $\triangle_B$ for any feasible $\triangle_B$. Then estimate $\rho_1\left(\triangle_B^{min}\right)$:

$$\rho_1\left(\triangle_B^{min}\right) = -\left(\triangle_A^-\right)^3\left(p_1 k^2 + p_2\right) > 0,\; p_{1,2} > 0$$

This implies $\rho_1\left(\triangle_B\right) > 0$ for any feasible $\triangle_B$. As a result, $\frac{\partial J^-}{\partial \triangle_B}\left(\triangle_A^-,\, \triangle_B\right)$ must increase in $\triangle_B$ for any feasible $\triangle_B$ and $\triangle_A^-$.

Next, I investigate $\frac{\partial^2 J^+}{\partial \triangle_B^2}\left(\triangle_A^-,\, \triangle_B\right)$:

$$\frac{\partial^2 J^+}{\partial \triangle_B^2}\left(\triangle_A^-,\,\triangle_B\right) = \frac{\rho_2 k^2 + \left(\triangle_A + 4\triangle_B\right)^3}{\left(3\triangle_A + 40\triangle_B\right)^3\left(\triangle_A + 4\triangle_B\right)^3}$$
$$\rho_2 = \sum_{j=0}^{3} x_{2j}\triangle_A^j\triangle_B^{3-j},\; x_{2j} > 0\,\forall\, j$$

Again, the denominator and the second element of the numerator are positive for all $\triangle_B \in \left[\tilde{\triangle}_B^n\left(\triangle_A^-\right),\, 1\right]$. Treating ρ_2 as the function of $\triangle_B$ delivers:

$$\frac{\partial \rho_2(\triangle_B)}{\partial \triangle_B} \le 0 \Leftrightarrow \triangle_B \in [-0.01,\, -0.11]\,\triangle_A^-$$
$$\rho_2\left(\triangle_B^{min}\right) = -p_3\left(\triangle_A^-\right)^3 > 0,\; p_3 > 0$$

and ρ_2 must be positive for any feasible $\triangle_B$. Hence, $\frac{\partial^2 J^+}{\partial \triangle_B^2}\left(\triangle_A^-,\,\triangle_B\right) > 0$ follows, and $\frac{\partial J^-}{\partial \triangle_B}\left(\triangle_A^-,\,\triangle_B\right)$ has to increase in $\triangle_B$ for any feasible $\triangle_B$ and $\triangle_A^-$.

Lemma 8 implies that a sufficient condition for $\frac{\partial J}{\partial \triangle_B}\left(\triangle_A^-,\,\triangle_B\right)$ to increase in $\triangle_B$ for any $\triangle_A^-$ is $\frac{\partial J^-}{\partial \triangle_B}\left(-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-,\,\triangle_B\right) > 0$ and $\frac{\partial J^+}{\partial \triangle_B}\left(\tilde{\triangle}_B^n\left(\triangle_A^-\right),\,\triangle_B\right) > 0$: □

$$\frac{\partial J^+}{\partial \triangle_B}\left(\tilde{\triangle}_B^n\left(\triangle_A^-\right),\,\triangle_B\right) = -\left(\triangle_A^-\right)^4\left(q_1 k^2 - q_2\right),\; q_{1,2} > 0,\; \frac{q_2}{q_1} > 1$$
$$\frac{\partial J^-}{\partial \triangle_B}\left(-\frac{\bar{\alpha}_1}{\beta_1}\triangle_A^-,\,\triangle_B\right) = \left(\triangle_A^-\right)^4\left(q_3 k^2 + q_4\right),\; q_{3,4} > 0$$

As a result, for $k \in \left[0,\, \min\left\{\tilde{k},\, \bar{k}\right\}\right)$ the objective function $J\left(\triangle_A,\,\triangle_B\right)$ monotonically increases in $\triangle_B$ for any feasible $\triangle_A$, and $\triangle_B = 1$ is indeed optimal.

To sum up, for the set of symmetric probability–valuation profiles $\bar{R}_k^L = \{0.5,\, 0.1,\, 0.15,\, 0.01,\, 2,\, 1,\, k\} \in R_k^L$ such that $k \in \left[0,\, \min\left\{\tilde{k},\, \bar{k}\right\}\right)$ the designer chooses $\triangle_A = -1$, $\triangle_B = 1$ and uses goods' endowments completely.

□

Chapter 2

Take Me to Court: Explaining “Victim-Defendant” Settlements under Asymmetric Bargaining Positions

Joint with Madina Kurmangaliyeva

2.1 Introduction

The state sets the rules to resolve civil and criminal disputes. It can stick to retribution or deterrence concerns, try to save on costs of the conflict intervention, but overall, the design of the justice system is a very complex issue.[1] The existing mechanisms of the dispute resolution are very diverse, especially in the criminal law. Some countries (for example, the United States) adopt a paternalistic approach and take the criminal conflict out from the victim. Other legal systems, such as Russia and a few other post-Soviet countries (Ukraine, Kazakhstan, Latvia, Lithuania), can delegate the dispute resolution to the parties involved. Disclosing strong and weak sides of each institutional setting is crucial for the optimal design of the justice system.

Generally, the state prefers to not intervene into the dispute resolution if the conflict does not threaten social interests. For example, opponents in civil lawsuits can always settle among themselves and go to court only if they fail to achieve an agreement. This practice allows the society save on costs of litigation.[2] There is a room for the state to delegate the conflict resolution in case of the criminal justice as well. Around 20% of European convicts are sentences for less than 1 year.[3] [4] At the same time, the average annual cost per inmate reaches €16'425 in Europe. Hence, making the criminal prosecution more flexible to minor-harm non-intentional crimes by means of "victim-defendant" settlements could help the state decrease its expenses on prisons.[5]

Nevertheless, the decentralized case resolution may also create some problems for the society. First, excluding the state from the conflict can result in milder sanctions and reduce deterrence. Another important concern relates to direct (violent) or indirect enforcement to settle.[6] The latter aspect occurs because of the power imbalance between the conflicting parties. If one individual displays a strong advantage, the opponent feels discouraged and reduces his fighting effort. Since winning the conflict becomes less likely, this player agrees on a lower settlement offer than he would request from a rival of the same strength. Hence, the settlement turns to be cheaper for individuals with better fighting abilities, and the power imbalance at the negotiation stage can increase the inequality before the law.

Explaining the settlement process under asymmetric bargaining positions is crucial for the optimal design of the justice system. This paper sheds light on key driving forces behind

[1] The costs of the conflict intervention include salaries for judges and prosecutors, expenses on prisons etc.

[2] If the case does not go to court, it reduces the load on judges. Also, the conflicting parties face no need to spend money for lawyers and waste their time on trials.

[3] See http://criminaljusticealliance.org/wp-content/uploads/2015/02/CJA_ReducingImprisonment_Europe.pdf

[4] In 69 countries, a number of inmates per 100'000 population exceeds 200 (see http://www.prisonstudies.org/highest-to-lowest/prison_population_rate).
21 European states, including Italy, Belgium, and France, had overcrowded prisons in 2013 (see http://www.telegraph.co.uk/news/uknews/crime/11405588/England-and-Wales-near-top-of-prison-spending-league-table.html).

[5] For example, Russia allows for "victim-defendant" settlements in case of criminal traffic offenses.

[6] The direct enforcement to settle takes place when victims face credible threats from the defendants' side.

out-of-court negotiations and emphasizes what kind of problems the institute of "victim-defendant" settlements may cause for the society. Specifically, we propose a stylized theoretical model with two players, the victim (she) and the defendant (offender, or he), who differ in their (dis)-utility of punishment, non-monetary fighting abilities and face asymmetric budget constraints. All individual characteristics are assumed to be common knowledge. The two conflicting parties exert costly effort in order to move the case to court. Before the fight starts, the defendant can make a "take-it-or-leave-it" offer to the victim. If the proposal is accepted, the case closes with a settlement. Otherwise, the players enter the contest game, and the defendant gets punished with a positive (endogenously defined) probability.[7]

In the absence of private information, the optimal settlement offer (S) must be equal to the victim's equilibrium payoff at the contest stage. The amount of S decreases when the bargaining position of the defendant strengthens (namely, his disutility of punishment rises or fighting abilities improve). More resources and higher winning benefits allow the defendant increase his effort and reduce the probability that the case ends up in court. Consequently, the victim's equilibrium payoff declines. Hence, being matched with a richer opponent does not guarantee the victim a better settlement offer. On the contrary, if the victim gets stronger, the amount of S grows, and it becomes more expensive for the defendant to avoid the punishment.

When the victim is not very vindictive (i.e. her winning benefit is less than the disutility of punishment the offender faces), the defendant always prefers to make an offer. In this sense, the settlement is efficient. However, if the victim extracts high vengeance benefits, even feasible settlements can fail to happen. In other words, the defendant who has sufficient resources to pay the amount of S decides to enter the contest stage. This scenario requires the victim to be strong enough, which makes the settlement relatively expensive. At the same time, the offender must display a sufficiently high probability to win the contest stage and avoid the punishment. If the defendant makes the settlement offer, the victim accepts the proposal, and the amount of S must be paid with probability 1. Since the defendant still has high chances to avert the court, entering the contest stage is associated with a greater expected payoff and becomes a more attractive alternative.

To recover the distribution of players' bargaining positions, we bring the model to the data on criminal traffic offenses in Russia. Here, defendants who have no previous felonies can settle with their victims out of court. We use a centralized police database that covers the period from 2013 to 2014 and aggregates criminal traffic offenses across 84 Russian regions (about 70'000 cases in total). The dataset contains detailed information about victims' and defendants' socio-economic characteristics, accident-specific controls and reports the case outcome. Around 17% of conflicts settle.

[7]We use the Tullock contest success function to model a probability to reach the court.

First, we provide the reduced form evidence that non-monetary fighting abilities matter for the conflict resolution. To identify the effect of interest, we focus on law enforcers and government officials. This group tends to know the justice system better and has well-established connections with legislators and other powerful individuals. Controlling for defendants' wealth (their expected car prices), law enforcers and government officials are more (less) likely to settle with their victims (end up in prison) than other offenders.

Further, we structurally estimate the model on two different subsamples. Overall, the proposed theoretical framework successfully replicates the observed probabilities to settle, to end up in court and to get a real sentence. On average, victims hold 10 times less wealth than defendants. Moreover, the degree of the resource imbalance becomes stronger when we focus on "car vs. pedestrian" matches. Victims who happen to be close relatives of their offenders are 9.6 times less vindictive than strangers (pedestrians). At the same time, the value of winning the case for each opponent positively depends on his / her wealth.

In the structural setting, we do not find the evidence that law enforcers and government officials display better non-monetary fighting abilities. This fact, however, can be explained with the selection bias. Specifically, very powerful law enforcers and government officials can use their influence to avoid the investigation stage. As a result, they never appear in the sample, and the estimator needs to be adjusted respectively.

We use the model to approximate expected equilibrium payoffs defendants get when they settle with their victims and end up in court. The former value turns to be significantly less than the contest stage payoff (–.007 versus –810.78 if we take "car vs. pedestrian" matches only). The disparity persists even when we divide the accidents into groups based on the harm made. Hence, those defendants who have enough resources to settle with their victims encounter much milder punishment, and this indeed generates the inequality before the law.

Further, we perform counterfactual experiments with the estimates obtained. In the beginning, we evaluate how the prison population changes if the state abandons "victim-defendant" settlements. As a result, 30% of the previously settled cases (more than 2'850 matches in the complete dataset) would close with a real sentence.[8] If all these individuals were convicted for one year, this would cost Russia €2.3 million.[9]

Then, we investigate how increasing the defendant's wealth (i.e. improving his bargaining position) affects the optimal offer and changes his ability to settle. Relaxing the offender's resource constraint has two effects. First, it makes him able to afford better offers and settle with more victim types ("volume effect"). Second, higher wealth improves the defendant's bargaining position and allows the player increase his effort. This reduces the victim's equilibrium payoff and drives

[8]The total number of cases with complete "victim-defendant" profiles amounts to 56'000.

[9]The calculation is based on €2.2 per day for one prisoner.
See http://www.rbc.ru/society/11/02/2015/54db24779a794752506f1ebf.

the optimal settlement offer down ("price effect"). Actually, the amount of S displays an inverse U-shape with respect to the offender's wealth. When the defendant is relatively poor and his resource constrain relaxes slightly, the "volume effect" prevails, and the average settlement offer rises. At some point, the offender becomes sufficiently rich, and the "price effect" gets stronger.

Finally, we perturb the wealth distribution for victims and trace the effect on their bargaining positions. With our identification restrictions, more resources available drive vindictiveness up, and eventually, victims systematically obtain higher winning benefits than their opponents. This increases the optimal settlement offer, and a probability to achieve a pre-court agreement declines for two reasons. Obviously, some defendants cannot afford the amount of S any longer. On top of this, we start observing offenders who have enough wealth to settle but prefer to enter the contest stage. To generate the latter result, the resource advantage must be on the victims' side, which is true for some types of crimes and civil disputes.

To complete the analysis, one must discuss "victim-defendant" settlements from the social welfare prospective. In our theoretical setting, the given institute makes no conflicting party worse off, and defendants gain much more than victims. Hence, with the utilitarian welfare function, the presence of pre-court agreements never hurts the society.[10] However, if the policymaker also cares about deterrence or has equality concerns, "victim-defendant" settlements may reduce social welfare because this legal practice leads to significantly milder sanctions for offenders with better bargaining positions. Overall, a more thorough discussion needs a particular social welfare function.

The paper proceeds as follows. Section 2.2 examines the related literature and indicates a contribution our work makes. Section 2.3 describes how Russian justice system processes criminal traffic offenses. Section 2.4 introduces the model and states our main theoretical results. Section 2.5 characterizes the data and a structural setup, reports estimation results and counterfactual experiments. Section 2.6 concludes.

2.2 Related Literature

This paper brings together two strands of the Law and Economics literature. In particular, our work contributes to research on settlements and relates it to the literature dealing with resource imbalances and unequal access to justice. The former field traditionally develops game-theoretic models of settlements and praises cost-saving benefits of this institute, both in civil litigations and the criminal law (plea bargaining). Here, trials are treated as a failure to achieve an agreement, and a considerable body of research has been trying to explore what provokes this "inefficiency" (Spier (2007)). Among the reasons, the literature cites asymmetric information (Reinganum (1986)), divergent beliefs of the parties (Landes (1971); Priest (1984)) and, for civil disputes, binding budget

[10]Here, the policymaker simply aggregates utilities over all victims and defendants.

constraints defendants may face. In this paper, we also investigate what explains settlements in case of criminal offenses. However, instead of looking at plea bargaining, we concentrate on "victim–defendant" negotiations and endow offenders with a right to make a "take-it-or-leave-it" offer.[11] We show that relaxing the defendant's budget constraint does not necessarily lead to more settlements and higher compensation amounts in equilibrium. On the contrary, the willingness to settle may vanish because the defendant can use the resources to increase his probability to win the trial instead. Hence, in our model, no settlement can not be interpreted as a negotiation failure: actually, going to court is optimal from the defendant's prospective.[12]

Despite their cost-saving benefits, the settlements also rise some objection. Usually, the criticism points out the increased coercion of guilty pleas from innocents (Langbein (1978); Alschuler (1981)) and the inability to reach socially desirable outcomes (Polinsky (1988); Garoupa and Stephen (2008)). One of the most prominent critics of civil case settlements in the law literature, Fiss (1983), argues that scholars usually assume no resource asymmetries between the parties involved. In reality, richer defendants may force their victims to accept lower offers, so the latter party still incurs implicit costs of litigation through a reduced settlement amount. According to Fiss (1983), this goes against the core idea of justice, which should prevent such distortions. Glaeser et al. (2003) perceives unequal outcomes of the justice system as a sign of the overall institutional deterioration. Also, it is broadly discussed whether rich defendants should be allowed to transform their resources into better legal counseling (Lott (1987); Garoupa and Gravelle (2003)).

In our paper, the power imbalance plays a crucial role. We construct a theoretical model and propose an empirical framework to study the impact of asymmetric preferences and fighting abilities on the outcome of the settlement stage. Indeed, defendants with weaker resource constraints and / or better non-monetary characteristics (for example, well-connected individuals) manage to avoid the court at a lower price. On top of this, we approximate the inequality in judicial outcomes for settled and non-settled cases and run counterfactual experiments to trace the effect of banning pre-court agreements on the prison population.

The paper also contributes to the literature on victims' role in the criminal process, which usually builds upon the idea of restorative justice (Zedner (2011)). In this case, the state takes the conflict away from the victim who becomes a passive actor (Sebba (1996); Shapland (2000); Strang (2003)).[13] The settlements we consider represent an alternative dispute resolution that brings the agency back to victims (Harland (1982)). Generally, this allows the victim reveal her taste for

[11] In case of plea bargaining, it is the prosecutor who makes an offer to the defendant, and the victim does not play any role.

[12] With our modeling assumptions, the victim stays indifferent between settling with the defendant and going to court.

[13] The literature on plea bargaining follows this idea and never looks at the victim's side.

vengeance (Glaeser and Sacerdote (2003)). Using our identification approach, we estimate how vindictiveness varies across different demographic groups and types of harm for Russia.

Our work is not the first one to structurally estimate the model of settlements. The most recent study is Silveira (2017). The paper focuses on plea bargaining under asymmetric information and proposes a non-parametric estimator to recover the distribution of defendants' types (their probabilities to be found guilty). Merlo and Tang (2016) look at civil settlements in medical malpractice disputes and recover beliefs of the conflict participants. As the authors claim, a failure to reach a pre-court agreement may arise from excessive optimism of the parties involved. Merlo and Tang (2016) find that the plaintiff's perception of winning the trial changes with the harm made and the identity of his opponent (in this case, a doctor). Sieg (2000) and Watanabe (2006) also employ the data on medical malpractice litigations. The former paper shows that the bargaining model with settlements replicates all observed patterns quite well. Watanabe (2006) studies dynamic aspects of the negotiation process and emphasizes the role of learning about the opponent's beliefs in achieving the settlement.

This work differs from the aforementioned studies in several respects. First, we concentrate on "victim-defendant" settlements in the criminal law. Second, imperfect information concerns are left out, and the research focus shifts to resource asymmetries. Third, we build upon a different model. Now, if the settlement does not happen, the case ends up in court with a certain probability that depends on efforts conflicting parties exert. Fourth, we do not observe settlement offers, which was not the case in the previous studies. However, our theoretical framework and case-specific controls available allow us build a parametric estimator and recover the distributions of players' preferences and fighting abilities.

Finally, the modeling approach we develop in this paper brings us closer to the literature on contests and conflicts. In particular, the "victim-defendant" interaction is treated as a Tullock-type competition where both parties exert costly effort in order to reach / avoid the court stage. Okuguchi and Szidarovszky (1997) study an asymmetric Tullock contest and prove the existence of a unique pure strategy Nash equilibrium. Yamasaki (2008) extends their result by adding player-specific budget constraints and focusing on a very general contest success function. Baye at al. (1994) analyze a discrete Tullock rent-seeking model with two homogeneous players and a contest success function that displays increasing returns to scale. In this class of games, the equilibrium cannot be derived from first-order conditions. The authors, however, prove that a symmetric equilibrium in mixed strategies exists and develop an algorithm to construct it. We use the indicated results to analyze contestants' decision to settle among themselves, which is a key issue our model aims to explain.

The game we design is essentially a conflict, and this research field builds upon a contest mechanism. Sambanis et al. (2017) analyze a fight between two groups, the government and the rebels, under a threat of an external intervention. The interaction is modeled as a Tullock-type contest where the rebels can make the government a settlement offer. However, this study does not take account of asymmetric resource constraints and in this respect differs from the model we propose.[14]

Robson and Skaperdas (2008) consider different ways to resolve conflicts over property rights. They employ a dynamic contest setting with two players. The parties can either settle or go to court, but both options are associated with positive enforcement costs. The authors show that going to court immediately can be preferred to the settlement. The result is driven by contestants' willingness to save on enforcement costs and avoid discounting. Our model is static and accommodates only the cost of exerting effort. Nevertheless, we show that the defendant can strictly prefer to go to court even if the settlement does not require any additional cost. This finding strongly relates to contestants' heterogeneity in preferences and fighting abilities.[15]

2.3 A Legal Process for Criminal Traffic Offenses in Russia

According to Russian laws, all traffic offenses are classified into civil and criminal cases. The accident enters the latter group if it resulted in serious bodily injuries, which must be certified by the forensic medical exam results.[16] The Criminal Code of Russia categorizes respective traffic offenses based on a number of fatalities (namely, no death, one death, and multiple deaths). Also, it distinguished between sober and drunk drivers. The combination of these two characteristics defines six types of criminal traffic accidents. The highest possible prison sentence changes with the offense category. For example, a driver of "no death & sober" type can get at most two years of incarceration. At the same time, an offender from "multiple deaths & drunk" group may spend up to nine years in jail. On top of prison sentences, drivers can temporary lose his licenses.[17] Also, the court decides how much the defendant must pay in order to cover all moral damages the victim faced. The compensation of medical expenses and property damages is determined by civil courts, and this usually involves insurance companies.

[14] Another example where the conflict analysis builds on a Tullock-type contest without budget constraints is Esteban and Ray (2011).

[15] Baye et al. provide one more application of the contest theory to the legal setting. The authors employ the all-pay auction model to analyze symmetric litigation disputes and compare aggregate legal expenditure under different institutional frameworks.

[16] According to the Criminal Code, bodily injuries can be classified into light, average, and serious ones for the purposes of prosecution. The division builds on the forensic medical exam results. According to the Code, a serious bodily injury must be "hazardous for human life" or involve the loss of sight, speech, hearing, or any organ or the loss of the organ's functions. Also, the legal definition accounts for a permanent loss of a general ability to work, an interruption of pregnancy, mental derangements, or post-traumatic addictions.

[17] In case of imprisonment, the license withdrawal starts the day after the offender's release from jail.

Assume a traffic accident happens. The police station that controls the location where the offense took place must register the case as a criminal one if there is at least one death or a medical report about serious bodily injuries. Then, the case goes to an investigator who collects and analyzes all pieces of evidence: medical certificates, witness testimonies, experts' reports, photographs and video materials etc. If the offender escapes after the accident, it is also the police job to find this person. By the end of the process, the investigator passes all the materials to a prosecutor.

Based on the evidence, the prosecutor decides whether to send the case to court.[18] At this stage, the defendant with no criminal history and the victim can settle in a civil case fashion and dismiss the criminal charge. In particular, the offender voluntarily compensates all moral damages the victim faced. The victim forgives the defendant and officially, in a written form, asks for the criminal prosecution to be stopped, subject to the approval of the investigator (with the permission of the prosecutor) or to the judge.[19] The offender gets no criminal record because his guilt has not been verified in court. However, the fact of the settlement enters all police databases and can be observed by external parties (for example, potential employers or other government entities).

If no settlement agreement was reached, the judge uses the evidence provided and decides on the defendant's guilt. One remarkable feature of Russian criminal system is that in-court acquittals are very rare (less than 1% out of all cases). If the defendant is found guilty, the judge may suspend the prison sentence.[20] For the "no death & sober" offense type, the judge may also replace a real incarceration term with different restrictions of liberty, which are milder than prison. It allows the offender live usual life, except for certain geographical limitations.

2.4 The Model

2.4.1 Model Setup

To characterize the interaction between the victim (V, or she) and the defendant (D, or he), we introduce a simple contest model with two heterogeneous players. Such a setup is commonly used in the conflict literature to represent situations where parties exert costly effort in order to win a battle.[21] In our instance, V fights against D for the case being considered in court. Once it happens, D can get recognized as guilty, and the punishment follows. Also, we depart from standard models in two ways. First, V and D can settle among themselves before entering the

[18]When the police identifies a deceased person as the offender, the case usually does not go to court and closes with conviction.

[19]If a true victim dies, his / her close relatives are recognized as victims.

[20]The sentence suspension applies only to first-convicted offenders. Otherwise, the judge must assign a real jail term.

[21]For example, see Esteban and Ray (2011), Sambanis (2017), Robson and Skaperdas (2008).

contest. Second, we introduce asymmetric budget constraints for the players, and this modification leads to a richer set of possible equilibria and non-trivial settlement decisions.[22]

Suppose an accident happens, and V and D are matched against each other. In the beginning, consider a so-called "in-court" scenario. Let p^h be a probability that D is found guilty.[23] Then, D gets punishment $x \geq 0$ and faces a total monetary disutility of $\{-bx\}$ where $b > 0$.[24] At the same time, V gains $\{ax\}$ in monetary terms, $a > 0$. We interpret a as V's vindictiveness and do not restrict how b and a relate to each other (both $a \geq b$ and $a < b$ are feasible).

Clearly, V and D have misaligned preferences. The "in-court" outcome is desirable for V (victim); however, D (defendant) would like to avoid this scenario. It results in a conflict where both V and D are willing to exert effort (e_V and e_D, respectively) and change the outcome in their favor. To model how a probability to end up in court (P_C) depends on players' effort choices, we employ a standard Tullock contest success function:

$$P_C\left(e_V,\, e_D\right) = \frac{e_V^r}{e_V^r + e_D^r},\; r = 1$$

Also, we state that if no party exerts positive effort, the case certainly goes to court, i.e. $P_C\left(0,\, 0\right) = 1$. This assumption is not standard in the literature; however, in case of criminal offenses, it makes perfect sense to break a "0–0" tie in the victim's favor.

Next, suppose V (D) has a total budget of $w_V \geq 0$ ($w_D \geq 0$), which can be spent on effort e_V (e_D). Also, define a player-specific cost parameter, m_i, $i = \{V,\, D\}$. Hence, the total monetary cost of exerting e_V (e_D) is $\{m_V e_V\}$ ($\{m_D e_D\}$).[25] We treat w_i and m_i, $i = \{D,\, V\}$ as monetary and non-monetary fighting abilities, respectively. The interpretation of w_i is quite intuitive: more resources can buy stronger lawyers who are able to build a high quality defense. Non-monetary fighting abilities reflect players' connections (for example, their access to or the position in the network of legislators etc.). In particular, lower m_i means that every monetary unit transforms into higher effort, and player i can fight more with the same total budget. We assume that contestants' utility is additively separable in punishment (x) and the cost of effort ($\{m_i e_i\}$, $i = \{V,\, D\}$). Finally, monetary and non-monetary fighting abilities, as well as contestants' preferences, constitute common knowledge.

[22]For example, D may not want to settle even if he has enough budget to do so.

[23]Generally, p^h must depend on the true state of guilt: those who are actually culpable are more likely to get a conviction. Since we concentrate on unintentional crimes (namely, traffic accidents), we simply the analysis and do not introduce guilty / non-guilty types separately. However, the model can accommodate the guilt-dependent likelihood of conviction easily.

[24]Generally, the punishment x is case-specific and depends on the level of harm made to a victim and the degree of quilt.

[25]Here, we work with a linear cost function. The analysis extends to the case of convex cost specifications.

At the contest stage, V and D choose their effort levels to maximize expected payoffs given budget constraints:

$$\begin{aligned}
V: \quad & \max_{e_V} \pi_V(e_V, e_D) \\
& s.t.\, \pi_V(e_V, e_D) = axp^h P_C(e_V, e_D) - m_V e_V \\
& m_V e_V \leq w_V,\, e_V \geq 0 \\
D: \quad & \max_{e_D} \pi_D(e_V, e_D) \\
& s.t.\, \pi_D(e_V, e_D) = -bxp^h P_C(e_V, e_D) - m_D e_D \\
& m_D e_D \leq w_D,\, e_D \geq 0
\end{aligned}$$

Now, we introduce a pre-contest stage where V and D can settle. Assume the defendant makes an offer S to the victim before entering the conflict phase.[26] [27] For simplicity, if S is such that the victim is indifferent between settling and fighting, she accepts the offer.[28] Further, we define contestants' bargaining positions.

Definition. Contestant i's bargaining position, $i = \{V, D\}$ is a combination of his / her (dis)-utility of punishment, monetary and non-monetary fighting abilities (w_i and m_i, respectively).

Overall, the game proceeds as follows:

1. Defendant (D) makes an offer S to Victim (V). If V accepts the proposal, the game ends. Otherwise, D and V move to the contest stage.

2. D and V simultaneously choose their effort levels, e_D and e_V, respectively.

3. The contest outcome realizes (the two parties either end up in court or the case closes), and the agents get their payoffs.

To solve the game, we proceed by backward induction.

2.4.2 The Contest Stage

When V and D do not manage to settle, they move to the contest stage. Proposition 2.1 provides a general equilibrium characterization of the contest game:

[26]As lawyers say, in most of the cases it is indeed the defendant who makes a settlement offer.

[27]In principle, one could model the pre-contest stage as a Nash bargaining game where V and D split the surplus among themselves. However, to identify contestants bargaining power, it is crucial to observe the settlement amount, which is never reported. For this reason, we stick to a simplistic assumption of D making a first move and extracting all the surplus.

[28]The analysis extends to the case when V can randomize between settling and fighting.

Proposition 2.1. *The equilibrium of the contest stage exists and is unique.*

Proof. See Appendix A. □

The existence and uniqueness results are proven by construction. In equilibrium, both contestants always stay active. When players' budget constraints do not bind, we get a standard asymmetric Tullock contest with two participants. This case is well-studies in the literature. With the given contest success function, the equilibrium is always interior and unique. Also, it features pure strategies. If only one constraint binds, a player with limited resources expends all the budget, i.e. $e_i = \frac{w_i}{m_i}$, $i = \{V, D\}$ becomes optimal. The opponent's best reply to $e_i = \frac{w_i}{m_i}$ solves his / her first-order condition and satisfies a feasibility requirement ($m_j e_j \leq w_j$, $j \neq i$). Here, the constrained player strictly prefers to stay active because only then he / she gets a chance to win against the advantaged opponent.

When both budget constraints bind, the players decide whether to exert positive effort ($e_i = \frac{w_i}{m_i}$) or abstain from participation ($e_i = 0$). In this case, the total effort cost contestants pay if choose $e_i = \frac{w_i}{m_i} > 0$ is always lower than the relative benefit of avoiding the punishment for D ($\frac{bxp^h}{m_D}$) / imposing the sanction on D for V ($\frac{axp^h}{m_V}$):[29]

$$\sum_{i=1}^{2} \frac{w_i}{m_i} < \min\left\{\frac{axp^h}{w_V}, \frac{bxp^h}{w_D}\right\}$$

Since a winner gains a lot compared to the cost paid, the competition is attractive for both players. Hence, V and D optimally select $e_i = \frac{w_i}{m_i} > 0$ and never abstain from participation.

Further, we summarize how contestants' equilibrium effort depends on the structure of their fighting abilities and the preferences over punishment.

Proposition 2.2. *Contestants' equilibrium effort, e_i^*, $i = \{V, D\}$ always increases in his / her valuation of punishment and w_i, decreases in m_i:*

$$\frac{\partial e_V^*}{\partial a} \geq 0,\ \frac{\partial e_D^*}{\partial b} \geq 0,\ \frac{\partial e_i^*}{\partial w_i} \geq 0,\ \frac{\partial e_i^*}{\partial m_i} \leq 0,\ i = \{V, D\}$$

For $\frac{a}{m_V} \geq \frac{b}{m_D}$

1. *e_V^* increases in b and e_D^* decreases in a*

2. *e_V^* decreases in m_D and increases in w_D*

[29] Here, we work with a rescaled version of the original contestants' programs where

$$\tilde{\pi}_V(e_V, e_D) = \tfrac{axp^h}{m_V} P_C(e_V, e_D) - e_V$$
$$\tilde{\pi}_D(e_V, e_D) = -\tfrac{bxp^h}{m_D} P_C(e_V, e_D) - e_D$$

This monotone transformation does not change the equilibrium V and D play.

3. *e_D^* increases in m_V and decreases in w_V if and only if $w_V \geq \frac{bxp^h m_V}{4m_D} > 0$. Otherwise, e_D^* strictly decreases in m_V and strictly increases in w_V*

For $\frac{a}{m_V} < \frac{b}{m_D}$

1. *e_V^* strictly decreases in b and e_D^* strictly increases in a*
2. *e_V^* increases in m_D and decreases in w_D if and only if $w_D \geq \frac{axp^h m_D}{4m_V} > 0$. Otherwise, e_V^* strictly decreases in m_D and strictly increases in w_D*
3. *e_D^* strictly decreases in m_V and strictly increases in w_V*

Proof. See Appendix A. □

Some results stated in Proposition 2.2 are straightforward. Players' equilibrium effort never decreases in the valuation they attach to the punishment. Higher a and b drive contestants' willingness to win up and make the competition more intense.[30] Also, better non-monetary fighting abilities (namely, lower m_i, $i = \{V, D\}$) decrease the effort cost and allow the players to fight more with the same budget. These two facts are well-documented in the contest literature. Other effects depend on both relative winning benefits ($\frac{a}{m_V}$ and $\frac{b}{m_D}$) and players' resources (w_D and w_V).

Take $\frac{a}{m_V} \geq \frac{b}{m_D}$ when V displays a stronger willingness to compete than her opponent. Here, the winning is relatively more desirable for the victim. Then, if D gets better stimuli to clash (b goes up or D's monetary and non-monetary fighting abilities improve), V wants to increase her effort as well and fights back. The opposite holds for the defendant. When a increases, the victim who already has an advantage ($\frac{a}{m_V} \geq \frac{b}{m_D}$) gets even stronger incentives to fight. This discourages D, and in equilibrium, he exerts less effort. The same happens if the victim has enough resources to expend ($w_V \geq \frac{bxp^h m_V}{4m_D} > 0$) and her monetary or non-monetary fighting abilities rise. However, the pattern reverts when V's budget constraint shrinks ($w_V \in \left[0, \frac{bxp^h m_V}{4m_D}\right)$).[31] In this case, D has enough monetary resources (compared to w_V) to overcome the victim's advantage and win. Similar logic applies when $\frac{a}{m_V} < \frac{b}{m_D}$, i.e. D displays a better bargaining position than V.

2.4.3 The Settlement Stage

In this subsection, we move one step back and analyze when V and D settle. Let π_i^*, $i = \{V, D\}$ be i's equilibrium payoff at the contest stage. First, we characterize how the optimal settlement offer S must look like.

[30] If a goes up, V extracts more utility from D being punished. Higher values of b translate into bigger costs of conviction for D, and his incentives to avoid the court stage increase.

[31] Although V has stronger incentives to win, she does not have a sufficient amount of money to support a desirable effort level.

Lemma 2.1. *The optimal settlement offer equals to V's equilibrium payoff at the contest stage, i.e.* $S = \pi_V^*$.

Proof. The proof is straightforward. Without loss of generality, suppose D's budget is unlimited, and he can afford any settlement offer. Also, assume D incurs significant losses in case of fight and is willing to avoid the contest stage. Formally, fix $\pi_D^* \ll -(\pi_V^* + \tau)$ where $\tau \gg 0$ is sufficiently high.[32] First, take $S = \pi_V^* + \varepsilon$, $\varepsilon > 0$ is small enough. The victim strictly prefers to accept the offer, and D's payoff becomes $\pi_D^{\varepsilon+} = -(\pi_V^* + \varepsilon)$. Next, consider $S = \pi_V^* - \varepsilon$. Now, the victim does not want to settle, the game proceeds to the contest stage, and $\pi_D^{\varepsilon-} = \pi_D^*$. Finally, check $S = \pi_V^*$. In this case, V accepts the proposal (see the assumptions of Subsection 2.4.1), and D gets $\pi_D = -\pi_V^*$. $S = \pi_V^*$ strictly dominates all other alternatives:

$$\pi_D = -\pi_V^* > \pi_D^{\varepsilon+} > \pi_D^{\varepsilon-}$$

and D prefers this strategy.

□

Lemma 2.1 illustrates a typical first-mover advantage. Since D makes a "take-it-or-leave-it" offer, he extract all the surplus in the absence of private information. If D prefers to avoid the contest stage ($\pi_D^* < -\pi_V^*$), proposing $S = \pi_V^*$ allows him to terminate the game, save on settlement costs and get the highest possible payoff.

Once the optimal settlement offer is defined, we check how it depends on the "victim–defendant" characteristics.

Proposition 2.3. *The optimal settlement offer S always decreases (increases) in D's (V's) willingness to win b (a) and his fighting abilities. S always increases in V's non-monetary fighting ability. S increases in w_V if and only if w_V is sufficiently small ($w_V \in [0, \tilde{w}_V]$, $\tilde{w}_V > 0$).*

Proof. See Appendix A. □

This result is quite intuitive. If the defendant gets stronger incentives to compete (either his winning benefit increases or fighting abilities improve), he exerts more effort. Depending on V's characteristics, the victim can either fight back or give up.[33] Under the former scenario, V faces higher effort cost; in the latter case, her winning probability decreases. Overall, V's equilibrium payoff declines, and it becomes easier to settle for the defendant.

[32] The extreme case would be $\pi_D^* = -\infty$.
[33] See Proposition 2.2 for more details.

The opposite happens when V's willingness to win grows or her ability to fight rises. In this case, D faces a stronger opponent who exerts significant effort, wins with a high probability and, consequently, obtains a larger equilibrium payoff. To prevent the fight, D must give the competitor a sufficient amount of money. Hence, settling with a mighty victim is more expensive (it may be even infeasible).

The effect of w_V on the optimal settlement offer depends on contestants' relative winning benefits (namely, $\frac{a}{m_V}$ and $\frac{b}{m_D}$). S becomes sensitive to w_V if and only if V's budget constraint binds, which happens for w_V small enough. Next, take the case of $\frac{a}{m_V} \geq \frac{b}{m_D}$ when V's relative utility from D being punished is sufficiently high. Then, if V's budget constrain binds, the optimal settlement offer S always increases in w_V.[34] This happens because V has a stronger willingness to win than her opponent. Hence, more resources allow the victim increase the effort, succeed with a higher probability, and obtain a better equilibrium payoff.

Further, assume $\frac{a}{m_V} < \frac{b}{m_D}$. Now, D has more incentives to win the contest and avoid the punishment. When V's budget constraint binds and w_V increases, two effects emerge. Obviously, the victim can fight more, i.e. e_V goes up. However, D also responds to growing w_V with higher effort.[35] In other words, the defendant, whose willingness to win is higher, does not feel discouraged when his opponent displays better monetary fighting abilities. With higher values of w_V, D engages into more fight, and at some point, V's winning probability starts decreasing. Also, the effort cost the victim must pay ($m_V w_V$) grows, and this coupled with lower values of $P_C(\cdot)$ drives V's equilibrium payoff down. Hence, the optimal settlement offer S declines in w_V for w_V sufficiently high because D competes more aggressively.

Proposition 2.3 implies that matching with a richer defendant does not lead to a better settlement offer (keeping V's characteristics constant). If w_D grows and D uses all his budget, the value of S must go down. The victim still accepts the offer made; however, her equilibrium payoff diminishes. This result goes against a conventional perception developed in the literature on "victim-defendant" settlements. The difference stems from our way to model the interaction between players. In particular, we use the contest framework where V and D challenge each other. Then, D's fighting abilities affect V's equilibrium payoff directly, and vice versa. The previous studies on the topic did not employ this competitive approach and could not discover the pattern we find here.

Overall, increasing w_D has two effects. For simplicity, take a population of potential victims. First, more resources allow the defendant settle with stronger opponents. In particular, he can afford the offers that were infeasible before. We call this the "volume effect". Second, those settlements that could appear even under lower values of w_D can happen with smaller offers.[36]

[34] See the proof of Proposition 2.3 for more details.

[35] See Proposition 2.2 for more details.

[36] If D's budget constraint did not bind in a particular match under lower w_D, the settlement offer does not

This pattern is labeled as the "price effect". Hence, more resources available make it easier for the defendant to avoid the fight not only because he can convince many victim types to settle, but also because it gets cheaper (the amount of S reduces).

Next, we analyze when the settlement indeed takes place. To prevent the conflict, two conditions must hold:

$$\pi_D^* \leq -S \Leftrightarrow -xp^h P_C\left(e_V^*,\, e_D^*\right)(a-b) + m_V e_V^* + m_D e_D^* \geq 0 \tag{2.4.1}$$

$$S \leq w_D \Leftrightarrow axp^h P_C\left(e_V^*,\, e_D^*\right) - m_V e_V^* \leq w_D \tag{2.4.2}$$

where asterisks denote equilibrium values. Condition (2.4.1) states that D must be willing to settle, i.e. his payoff from entering the contest stage cannot exceed the settlement cost. On top of this, the defendant has to hold enough resources to make an offer the victim would accept (inequality (2.4.2)). If at least one condition violates, the settlement does not happen. The first thing to notice connects players' preferences and D's willingness to settle. When V is not sufficiently vindictive (i.e. $a \leq b$), condition (2.4.1) always holds. In this case, the settlement is efficient. Otherwise, D may prefer to fight even if he has enough resources to make the offer required. Further, we concentrate on the latter case specifically.

Definition. Let $y = (a,\, b,\, m_V,\, m_D,\, w_V,\, w_D)$ be a "preference–abilities" profile, $y \in Y \equiv \mathbb{R}^6_{\geq}$.

Also, define $Y_{a>b}$:

$$Y_{a>b} = \{y \in Y \,:\, a > b\}$$

Proposition 2.4 illustrates that condition (2.4.2) not necessarily implies D's willingness to settle.

Proposition 2.4. *There exist non-empty sets of "preference–abilities" profiles $Y_{\bar{S}} \subset Y_{a>b}$ and $Y_S \subset Y_{a>b}$ such that*

- *For any $y \in Y_{\bar{S}}$ the defendant has enough resources to settle but is not willing to do so:*

$$\begin{cases} -xp^h P_C\left(e_V^*,\, e_D^*\right)(a-b) + m_V e_V^* + m_D e_D^* < 0 \\ axp^h P_C\left(e_V^*,\, e_D\right) - m_V e_V^* \leq w_D \end{cases} \neq \emptyset$$

- *For any $y \in Y_S$ the defendant has enough resources to settle and is willing to do so:*

$$\begin{cases} -xp^h P_C\left(e_V^*,\, e_D^*\right)(a-b) + m_V e_V^* + m_D e_D^* \geq 0 \\ axp^h P_C\left(e_V^*,\, e_D\right) - m_V e_V^* \leq w_D \end{cases} \neq \emptyset$$

Proof. See Appendix A. □

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change with w_D adjusting upward. Otherwise, S decreases with w_D. See Proposition 2.3 for details.

The result stated in Proposition 2.4 strongly relates to the victim's advantage (or disadvantage) in the contest. Whether feasible settlements always happen (namely, condition (2.4.2) implies (2.4.1)) also depends on whose budget constraint binds in equilibrium. Take the case when both contestants have enough resources to choose the interior effort level. Then, the defendant wants to settle if and only if V's non-monetary fighting ability is relatively low ($0 < \frac{1}{m_V} \leq \frac{2b^2}{am_D(a-b)}$).[37] Since both players are unconstrained and the victim turns to be vindictive enough ($a > b$), a single possibility D can dominate in the competition and drive the optimal settlement offer down comes from non-monetary fighting abilities (m_V and m_D). If V has an advantage in both a and m_V, in equilibrium, the amount of S must rise (Proposition 2.3), and the settlement becomes expensive. If the offer is accepted, D pays the value of S with probability 1. However, if the game proceeds to the contest stage, the defendant faces the punishment with probability less than 1, and his equilibrium payoff turns to be higher.[38] Hence, D prefers to fight even if he has enough resources to afford the settlement offer.

Next, take the case when only D's budget constraint binds. Here, the victim's advantage stems from both higher willingness to win ($a > b$) and more resources available. In this case, the defendant who has enough money to make the settlement offer always prefers to do so (condition (2.4.2) implies (2.4.1)). If the victim dominates in non-monetary fighting abilities as well ($m_V < m_D$), the optimal amount of S increases sufficiently. Then, the defendant who has limited resources can never afford the settlement offer and must proceed to the contest stage. When D has an advantage in non-monetary fighting abilities ($m_V > m_D$), he can compete more and drive the optimal amount of S down. Thus, avoiding the contest stage becomes feasible.[39]

If only V has limited resources, the case is similar to the unconstrained equilibrium we analyzed before. Again, V's characteristics affect the outcome of the settlement stage. When the victim dominates in non-monetary fighting abilities and / or incentives to win, this partly offsets D's advantage in w_D, and the optimal offer S rises. As a result, the defendant no longer wants to settle and prefers to move to the contest stage. However, if high values of w_D are couples with better non-monetary fighting abilities and / or stronger willingness to avoid the punishment (b), the settlement can happen.

Importantly, more wealth on D's side (w_D) not necessarily means that D is keen to settle. For example, take the case when only D's budget constraint is active. Also, assume w_D is sufficient

[37] The condition of interest can also be rewritten in terms of relative winning benefits:

$$m_V \geq \frac{am_D\,(a-b)}{2b^2} \Leftrightarrow \frac{a}{m_V} \leq \frac{b}{m_D}\frac{2b}{(a-b)}$$

[38] To make the contest more attractive than the settlement, D's equilibrium winning probability must be sufficiently high.

[39] Similar patterns appear when both contestants face binding budget constraints. See the proof of Proposition 2.4 for more details.

to make the optimal offer S. Then, feasibility (condition (2.4.2)) implies D's willingness to settle (condition (2.4.1)).[40] Now, increase w_D such that D's budget constraint does not bind and the resources are sufficient to make the new optimal offer S. If V has an advantage in non-monetary fighting abilities ($\frac{1}{m_V} > \frac{2b^2}{am_D(a-b)}$), the defendant is no longer willing to settle even if he holds enough budget. Here, making D stronger in terms of resources available does not offset the victim's dominance in winning benefits ($a > b$) and m_V. Specifically, the optimal settlement offer S does not reduce much. However, more money available allows the defendant increase his effort and avoid the punishment with a higher probability. The latter effect prevails, and the fight turns to be more attractive for D. Hence, those defendants who do not manage to settle not necessarily fail to meet the feasibility requirement (condition (2.4.2)): they can just display no willingness to avoid the contest stage.

The case where the defendant prefers to fight corresponds to relatively high aggregate equilibrium effort. Also, it features V's dominance in winning benefits and fighting abilities. In other words, victims win more often, and non-settled cases end up in court with a higher probability.

It is an open question whether no willingness to settle, combined with a sufficient amount of resources available to the defendant, can create any problems for the society. If the institute of "victim-defendant" settlements aims to delegate the case resolution to the parties involved when offenders have enough money to compensate their opponents and reduce the total load on courts, this goal might not be achieved. As emphasized earlier, this type of non-settled matches also features significant aggregate effort. It may translate into longer trials and higher processing costs for prosecutors and judges as well. Further, victims, who exert more effort in the given scenario, pay an additional cost on top of the harm they have already encountered. This observation drives us to revictimization concerns. Overall, the cases with feasible but not desirable settlements may generate some cost for the society, and we leave a broader discussion for the future.

2.5 The Empirical Analysis

In this section, we bring the proposed theoretical model to the data on criminal traffic offenses in Russia. As it was emphasized earlier, for this group of crimes, Russian laws allow defendants settle with their victims before entering the court stage. Also, traffic offenses constitute unintentional crimes where two conflicting parties are matched randomly. We exploit this feature in our identification strategy, structurally estimate the model, and highlight which channels explain the settlements observed.

[40] See the proof of Proposition 2.4 for more details.

2.5.1 The Data

2.5.1.1 Data Sources

To estimate the model, we use centralized databases that aggregate police-level data across 84 Russian regions for the period from 2013 to 2014. All investigators must fill in special statistical cards, which contain the information about different stages of the process.[41] The first database represents the universe of criminal traffic offenses that have been registered by police stations.[42] Here, a unit of observation is a case. The information available includes:

1. The time and the date when the accident happened (Form 1);[43]

2. The aggregate data on victims such as a number of deaths and / or serious bodily injuries, average bodily injuries plus the employment status of up to two victims (Form 1);[44]

3. The outcome of the investigation stage (Form 3).

Another database incorporates information about offenders' characteristics. Now, a unit of observation is a defendant. We observe:

1. The data on defendants' demographic attributes such as gender, his / her socio-economic status etc. (Form 2)

2. The defendant's history of criminal and administrative records (Form 2);

3. The court outcome, including the type of punishment and its duration (Form 6).

For every registered case, there can be no defendant (an offender has not been caught or did not get an accusation), exactly one defendant, and more than one defendant (the crime was committed by a group). Using the case identifier, the code of the police department, and the year when the accident happened, we merge the two datasets. Overall, 56′000 records have at least one defendant.

The third database provides detailed information for each victim. It includes:

1. Gender, which age and ethnic group the victim belongs to (Form 5);

2. The victim's employment status (Form 5);

3. His / her citizenship and residency (Form 5);

[41] The Institute for the Rule of Law at the European University at Saint Petersburg has an access to police-level statistical cards. This information is provided for research purposes under a restricted user agreement.

[42] We exclude the cases with military defendants because they are considered under the jurisdiction of military courts.

[43] All form numbers are set by the federal law.

[44] Unfortunately, the form does not allow us distinguish who, out of the two victims, got more severe injuries.

4. The harm caused by the offense (Form 5).

Once we merge this information with the first database, 57′000 cases have at least one entry from Form 5. However, some crimes stay unmatched. One possible explanation comes from investigators' behavior. Since Form 5 partly duplicates Form 1, they may skip this card in order to save time. As a piece of supportive evidence, we find a positive correlation between a probability of Form 5 missing and a number of victims reported in Form 1. Also, the absence of Form 5 displays a weak positive correlation with the outcome of the investigation stage, especially if investigators or prosecutors decided not to press charges.

The data also include a so-called *fabula* that describes the case shortly. Often, investigators use this document for their own easy reference. The description style and the amount of details it contains show significant variation across police departments. Usually, the *fabula* consists of two parts. First, it provides general information on the situation (time, location, weather conditions etc.) and the participants starting from the description of the offender's actions. As a rule, the text specifies the types of cars driven by the defendant and the victim (where applicable). It also mentions whether pedestrians were involved. The second part of the *fabula* describes the harm made and clarifies who the victim is: a pedestrian, a passenger, or a driver.

Using the information on cars mentioned in the *fabula*, their expected prices are imputed. To approximate these values, we collect data on prices of same-brand second-hand cars posted on *https://auto.ru/* in October, 2014.[45] Then, the first car mentioned in the *fabula* is attributed to the offender. We assign the second vehicle that appears in the *fabula* to the victim if and only if

- It is mentioned in the first part of the text and
- The second part explicitly attributes the victim to this car.

The information from all the *fabulas* is automatically processed with the use of regular expressions.

2.5.1.2 Descriptive Statistics

The dataset includes more than 70′000 registered criminal traffic offenses. This covers all 84 Russian regions, 2′500 police departments, and 700 courts.[46] Around 14′000 cases have no offender identified. Partly, it explains with hit-and-runs. On top of this, the accidents that took place in the end of 2014 were still under investigation when the dataset was collected. Finally, the police does not press charges for some of the identified offenders, and the prosecutors happen to drop cases as well.

[45] *https://auto.ru/* is one of the largest on-line platforms for private car sales in Russia.

[46] The data on courts are incomplete because around a half of the processed cases have no court identifiers.

In total, charges are pressed in 55′000 out of 70′000 cases. The information on victims is available for almost all registered criminal traffic offenses. Offenders' characteristics become observable only if the charges apply. We restore car prices for roughly a half of the offenders. At the same time, the information on vehicles where the victim has been injured is available only for a small subset of cases. Table 2.1 summarizes all the data available.

Table 2.1: Descriptive Statistics of The Data

Number of	
police departments	2′533
courts	705
regions	84
Number of cases	
by stage:	
case registered	73′661
offender is identified	59′868
offender is charged by the police	56′010
offender is charged by the prosecutor	55′240
by information available:	
info on victims	72′294
info on car prices for victims	5′904
info on offenders	56′280
info on car prices for offenders	29′777
by reporting period:	
2013	37′327
2014	36′334

We sort the variables into four blocks:

1. Characteristics of the accident and the harm made;
2. Victim-specific details;
3. Offender's characteristics;
4. The case outcomes.

Table 2.2 provides descriptive statistics for all the groups. The first block of observations includes a total number of victims and specifies how many of them ended up dead or seriously injured. The data also distinguish female victims and minors. Further, block (1) indicates whether the offender and the victim were intoxicated. Using the information from the *fabulas*, we recover the cases that involve pedestrians and passengers.[47]

[47] Any match with the phrase “hit a pedestrian” and its variations raises the flag for the variable *pedestrian*. Any match with the word “passenger” and its variations raises the flag for the variable *passenger*.

Table 2.2: Summary Statistics for Case-Specific Characteristics

Accident and Harm

variable	mean	sd
Number of victims	1.18	0.76
out of which:		
survived in the accident	0.66	0.59
died in the accident	0.45	0.70
minors	0.12	0.35
females	0.46	0.59
Under influence:		
offender	0.22	0.41
victim	0.04	0.19
Victim's role:		
pedestrian	0.27	0.44
passenger	0.26	0.44

Victim

variable	mean	sd
Female	0.44	0.50
Age:		
1 to 13 (8 y.o.)	0.07	0.25
14 to 15 (15 y.o.)	0.02	0.16
16 to 17 (17 y.o.)	0.03	0.17
18 to 24 (20 y.o.)	0.14	0.35
25 to 29 (27 y.o.)	0.12	0.33
30 to 49 (40 y.o.)	0.31	0.46
50 to 54 (52 y.o.)	0.07	0.26
55 to 59 (57 y.o.)	0.07	0.25
$\geq$ 60 (65 y.o.)	0.15	0.36
Employment status:		
no job	0.46	0.50
worker	0.24	0.43
office worker	0.02	0.15
top-manager	0.00	0.06
entrepreneur	0.01	0.10
budget office worker	0.01	0.12
student	0.08	0.27
welfare recipient	0.03	0.16
retired	0.13	0.34
other	0.01	0.08
In law enforcement:	0.01	0.10
Related:		
acquaintance	0.06	0.24
cohabitant	0.00	0.06
family	0.01	0.10
close family	0.03	0.16
Imputed car price (rub. mln)	0.31	0.26

Offender

variable	mean	sd
Female	0.09	0.28
Age:		
16 to 17 (17 y.o.)	0.01	0.08
18 to 24 (20 y.o.)	0.21	0.41
25 to 29 (27 y.o.)	0.21	0.41
30 to 39 (35 y.o.)	0.26	0.44
40 to 49 (45 y.o.)	0.15	0.36
50 to 59 (55 y.o.)	0.11	0.32
$\geq$ 60 (65 y.o.)	0.05	0.22
Employment status:		
no job	0.41	0.49
worker	0.42	0.49
office worker	0.03	0.18
top-manager	0.01	0.10
entrepreneur	0.03	0.17
budget office worker	0.02	0.14
student	0.03	0.17
welfare recipient	0.00	0.05
retired	0.04	0.20
other	0.01	0.08
In law enforcement	0.02	0.12
Education:		
college (16 years)	0.19	0.39
vocational (13 years)	0.34	0.47
technical (13 years)	0.02	0.14
high school (11 years)	0.35	0.48
secondary school (9 years)	0.08	0.27
elementary school (4 years)	0.01	0.12
no school (0 years)	0.00	0.05
Imputed car price (rub. mln)	0.29	0.25
Past offences:		
criminal history	0.17	0.38
administrative fines	0.10	0.30

Outcomes

variable	mean	sd
Settlements	0.17	0.38
In court	0.58	0.49
out of which:		
incarcerated	0.13	0.33
no information	0.24	0.43

Note: Summary statistics are provided only for the cases where offenders were charged by a prosecutor.

To make the analysis simpler, we use the information only on the first victim mentioned in Form 5; if this is not available, we exploit Form 1.[48] One can observe the victim's gender, his/her age group and employment status (see Table 2.2).[49] Additionally, we create a dummy to distinguish those individuals who work in law enforcement or in the government. Also, the data indicate whether two parties of the conflict know each other. In particular, a victim can be an offender's acquaintance, cohabitant, family member, or close relative. Finally, for some victims, we manage to restore expected car prices.[50]

For offenders' personal characteristics, the dataset is a bit richer. On top of socio-economic aspects, it provides information on offenders' educational background, which ranges from no education to holding a college degree (seven categories in total). All other demographic characteristics are the same as for victims, except the age groups do not coincide.[51] Also, we trace if the offender has any past criminal and/or administrative offense records.

As for the outcomes, the cases can be broadly categorized into:

1. Those that settled out-of-court,
2. Those that reached the court stage, and
3. Those that neither settled nor reached the court stage.[52]

If the case ends up in court and the offender is recognized as guilty, he can get a real incarceration term, receive a suspended sentence or face other forms of punishment. Some cases have missing outcomes. Most of such offenses were registered in the end of 2014 and were still at the investigation stage when the observation terminated. For simplicity, we treat these cases as those that have not reached the court yet.[53]

2.5.2 Non-Monetary Fighting Abilities: The Reduced Form Evidence

To illustrate that not only monetary resources affect the case outcome, consider some reduced form evidence. We focus on law enforcers and government officials. Belonging to these socio-economic groups has two non-monetary returns. First, law enforcers and government officials know

[48] When there are many victims, the first person registered is assumed to be the one with the least disputed victim status.

[49] The original data specify which age group a victim belongs to. Instead of creating a set of indicators, we recode the variable by taking the mean for every interval.

[50] Apparently, this measure displays a significant noise when used to approximate the victim's wealth. In fact, being a passenger of a certain car gives less information about one's income than driving this particular vehicle. To provide a robustness check, we will also estimate the model without this wealth proxy.

[51] This is the case because in Russia, criminal responsibility starts from the age of 14.

[52] The last group includes only the cases that had an indicted offender but were dropped later. We also acknowledge that there can exist criminal traffic offense that did not reach the sample, and they may constitute a sufficient share of all the cases. However, for now we do not account for this possibility.

[53] In the future, we are going to impute expected outcomes for such cases as a part of the estimation.

the institutional setting better and can defend themselves more efficiently in case of committing a crime / becoming a victim. Second, these people are connected to the networks of lawyers, legislators, and other mighty individuals. Hence, they may exploit the latter channel to affect the case outcome. For these reasons, we expect the given group to display different patterns.

Consider the following regression equation:

$$y_i = \alpha + \beta_D lawenf_D^i + \beta_V lawenf_V^i + \gamma lawenf_D^i lawenf_V^i + \psi_1 pcar_D^i + \psi_2 police_i + \psi_3 t_i + u_i \quad (2.5.1)$$

where

- $lawenf_l^i = 1$ specifies whether $l = \{V,\ D\}$ is a law enforcer or a government official (otherwise, $lawenf_l^i = 0$);
- $pcar_D^i$ reflects a mean car price for D;
- $police_i$ identifies a fixed effect of the police department;
- t_i captures year-specific effects.

Including $pcar_D^i$ into (2.5.1) allows us isolate the effect of $lawenf_l^i$, $l = \{V,\ D\}$. In particular, we compare law enforcers and government officials with individuals of the same wealth level but from other socio-economic groups. Estimation results are reported for three different samples. First, we consider all criminal traffic offenses available. Then, we focus only on "car vs. pedestrian" accidents where victims and offenders are definitely stranger.[54] Finally, to make the evidence even more convincing, we exclude low-status defendants (namely, unemployed individuals and welfare recipients) from the sample. Now, the baseline group – non-officials – becomes more comparable to law enforcers and government officials in terms of wealth. Standard errors are clustered at the police department level because a share of law enforcers and government officials is likely to vary across different locations.

Table 2.3 summarizes the estimates of β_D and γ specified in (2.5.1). The first three columns include all observations available; columns 4–6 report the results only for "car vs. pedestrian" cases; the last subsample (columns 7–10) also disregards low-status defendants (individuals without permanent job and welfare recipients). The "None" specification (columns 1, 4, and 7) does not control for car prices or brands.[55] The "Price" approach (columns 2, 5, and 8) and the "Brand" model include imputed car prices and brands as additional explanatory variables, respectively.

Now, we comment on the estimation results briefly. Controlling for defendants' wealth proxies, law enforcers and government officials are more likely to settle with their victims. Notice that the effect disappears when we estimate the model on the full sample (Table 2.3, columns 1–3): the

[54] With this approach, we eliminate all possible correlations between V's and D's wealth.

[55] Car brands identify trucks, buses, and motorcycles as separate categories.

Table 2.3: Case Outcomes When Defendants Are Law Enforcers and Government Officials

Sample:	All			Pedestrians†			Pedestrians & No low-status offenders‡		
Car controls:	None (1)	Price (2)	Brand (3)	None (4)	Price (5)	Brand (6)	None (7)	Price (8)	Brand (9)
SETTLED									
β_D	.003	.028	.029	.083	.216*	.217*	.069	.163*	.172*
	(.014)	(.028)	(.026)	(.047)	(.086)	(.086)	(.050)	(.081)	(.081)
γ	.158**	.142	.187	-.169	-.156	-.190	-.244	.012	-.073
	(.057)	(.105)	(.099)	(.122)	(.100)	(.102)	(.225)	(.117)	(.111)
N all	50987	21939	26266	11455	5581	6508	6771	3290	3860
	AMONG THE NON-SETTLED CASES								
REACHED THE COURT									
β_D	-.028	-.042	-.036	.041	-.066	-.060	.029	-.069	-.060
	(.020)	(.036)	(.034)	(.047)	(.104)	(.099)	(.060)	(.114)	(.110)
γ	-.023	-.071	-.101	-.033	-.006	-.010	-.085	-.117	.074
	(.065)	(.135)	(.124)	(.083)	(.128)	(.119)	(.101)	(.150)	(.141)
INCARCERATION OR DEPRIVATION OF FREEDOM									
β_d	-.064***	-.075*	-.062*	-.007	-.177*	-.146	-.015	-.157*	-.137
	(.018)	(.030)	(.028)	(.058)	(.073)	(.078)	(.070)	(.079)	(.084)
γ	.012	.149	.090	.098	.744***	.825***	.009	.107	.558***
	(.060)	(.122)	(.112)	(.195)	(.111)	(.115)	(.161)	(.151)	(.152)
INCARCERATION									
β_d	-.027	-.032	-.030	-.039	-.149*	-.111	-.023	-.126	-.099
	(.015)	(.024)	(.022)	(.042)	(.065)	(.070)	(.048)	(.068)	(.074)
γ	-.050	.021	.001	.067	.753***	.743***	.143	.729***	.799***
	(.051)	(.097)	(.089)	(.191)	(.097)	(.102)	(.235)	(.116)	(.118)
N not settled	42383	18091	21751	9485	4594	5373	5452	2636	3105

Note:
† Sample with one victim who is a pedestrian, not a student or retired;
‡ Same as † excluding low-status offenders (unemployed individuals and welfare recipients);
The regression with police department fixed effects; standard errors (in parentheses) are clustered at the police department level. To compute a number of settled cases, we use the lawsuits with non-missing information on victims' and defendants' employment status, which roughly matches the sample of cases with prosecutorial charges. The other outcomes – those that reached the court and where offenders got incarcerated or faced the deprivation of freedom – are based on the sample of non-settled cases.
Car brands identify trucks, buses, and motorcycles as separate categories.
See (2.5.1) for the meaning of β_D and γ.

null hypothesis of $\beta_D = 0$ cannot be rejected. At the same time, the interaction term γ is positive and statistically significant. It happens because the full sample accommodates all the cases where the victim got injured inside the defendant's car, and this implies non-random matching between the two conflicting parties.[56] Focusing only on "car vs. pedestrian accidents" (Table 2.3, columns 4–9) gives a clearer prediction. Controlling for car prices or brands, a probability to settle for law enforcers and government officials exceeds its counterpart for the baseline group by 20 percentage points. Excluding low-status defendants from the analysis reduces the magnitude of the effect (β_D gets smaller) but does not harm its statistical significance. The interaction term γ is negative, although insignificant.

Then, we check what happens with non-settled cases. The probability to reach the court does not display any variation across the groups: neither β_D nor γ are statistically significant in the respective regressions. However, the probability to get a strict punishment (namely, a deprivation of freedom or a real prison term) tends to differ. In particular, law enforcers and government officials are less likely to end up with a real sentence (β_D is negative and statistically significant). The effect is not so pronounced if we look at incarceration rates separately. Remarkably, the interaction term in the "car vs. pedestrian" sample is positive and quite large. These observations call for the following story. Suppose a law enforcer or a government official hurts a person from the same socio-economic group. If these individuals happen to know each other, they prefer to settle. However, if the two parties are strangers, the case is likely to move to the court stage where the defendant has worse chances to avoid the prison.

Overall, law enforcers and government officials tend to behave differently in the case resolution. Controlling for wealth proxies and non-random matching does not eliminate the discrepancies observed across groups, and we can indeed connect these patterns to non-monetary channels of influence.

2.5.3 The Structural Setup

To identify contestants' bargaining positions, we use the variation in the harm made, case outcomes, match- and individual-specific characteristics. The most important proxies for players' wealth are socio-economic status and imputed mean car prices. Also, the non-intentional nature of traffic offenses allows us avoid any issues related to self-selection into crime. With the proposed theoretical model, we formulate three key results and build the identification strategy upon them. [57]

Result 1. The optimal offer S increases in the expected punishment xp^h.

[56] For example, the victim and the defendant may be friends or colleagues who have the same occupation.
[57] See Section 2.4 for technical details.

In the data, the likelihood of observing a settlement indeed declines when the harm made (and, consequently, expected punishment) grows. The model relates this fact to increasing optimal offers that most defendants cannot afford. Hence, the variation in the observed harm can help us identify budget constraints offenders with comparable case- and individual-specific attributes face.

Result 2. The optimal offer S increases in the vindictiveness parameter a.

Matching with a more vindictive person makes a settlement more expensive, and the likelihood of reaching an out-of-court agreement decreases. Notice that the indicated effect works in the same direction as the one presented in ***Result 1***. To distinguish these two channels, we treat "Car vs. pedestrian" accidents with purely random matches as a control group for criminal traffic offenses where two sides of the conflict know each other. The latter subset includes more than 10% of the victim population (see Table 2.2). Looking at a particular level of the harm made, we can already see that settlements are less frequent for "Car vs. pedestrian" accidents and almost never happen for offenses with more than one death. The proposed identification scheme requires two assumptions to produce consistent and unbiased estimates:

1. The vindictiveness parameter a does not depend on the harm made;

2. Judges do not internalize the effect of non-random matching, i.e. the expected punishment does not correlate with victim-defendant relationship.

Result 3. Non-monetary fighting abilities and contestants' preferences can be separately identified.

For this claim to hold, one must observe individual wealth or have strong proxies for this variable. A separate identification of preferences and non-monetary fighting abilities becomes possible for two reasons. First, our theoretical model accommodates budget constraints. Second, the dataset includes uneven victim-defendant matches where these constraints are likely to bind.[58] Otherwise, one could not distinguish the effect of a (b) from m_V (m_D) without additional assumptions.

To illustrate ***Result 3***, consider a simple "2×2" example. Take a population of potential victims ($v = \{1, ..., N\}$) and defendants ($d = \{1, ..., N\}$) who match at random. Suppose there are only two wealth levels that are perfectly observed by an econometrician, i.e. $w_i = \left\{\underline{w}^i, \overline{w}^i\right\}$, $i = \{V, D\}$.[59] Further, assume player i's budget constraint always (never) binds when he (she) faces $w_i = \underline{w}^i$ ($w_i = \overline{w}^i$). For simplicity, we require all contestants to have identical preferences and non-monetary fighting abilities that the econometrician must infer:

$$a_v \equiv a,\, m_{V,v} = m_V \,\forall\, v = \{1,\, ...,\, N\}$$

[58]For instance, we observe individuals, who drive expensive cars, matched against people without a permanent job.

[59]The result still holds if contestants' wealth is observable with a small noise.

$$b_d \equiv d,\, m_{D,d} = m_D \,\forall\, d = \{1,\, ...,\, N\}$$

Let $f_{(w_D,\,w_V)}$ denote an empirical frequency of observing the $(w_D,\, w_V)$ match in court, and assume the following:

$$f_{(\underline{w}^D,\,\underline{w}^V)} = f_{\left(\overline{w}^D,\,\overline{w}^V\right)} = f_1,\; f_{\left(\overline{w}^D,\,\underline{w}^V\right)} = f_2 \,and\, f_{\left(\underline{w}^D,\,\overline{w}^V\right)} = 1 - f_2$$

Using the theoretical model developed in Section 2.4, we can compute case-specific probabilities to end up in court:

	$w_V = \underline{w}^V$	$w_V = \overline{w}^V$
$w_D = \underline{w}^D$	$\frac{\widetilde{w}_V}{\widetilde{w}_V + \widetilde{w}_D}$	$1 - \sqrt{\frac{\widetilde{w}_D}{\widetilde{a} xp^h}}$
$w_D = \overline{w}^D$	$\sqrt{\frac{\widetilde{w}_V}{\widetilde{b} xp^h}}$	$\frac{\widetilde{a}}{\widetilde{a} + \widetilde{b}}$

where $\widetilde{w}_i = \frac{w_i}{m_i}$, $i = \{V,\, D\}$, $\widetilde{a} = \frac{a}{m_V}$, and $\widetilde{b} = \frac{b}{m_D}$. Matching these theoretical moments against their empirical counterparts, we can obtain the following estimates:

$$\hat{a} = \frac{f_1 \underline{w}^V}{(1 - f_1)\, f_2^2} \frac{1}{xp^h},\; \hat{b} = \frac{(1 - f_1)\, \underline{w}^D}{f_1 f_2^2} \frac{1}{xp^h} \,and\, \widehat{\left(\frac{m_V}{m_D}\right)} = \frac{(1 - x_1)}{x_1} \frac{\hat{a}}{\hat{b}}$$

where xp^h can be identified by exploring the variation in the harm made and the corresponding court decisions. Unfortunately, we cannot separate the effect of m_V from m_D without imposing additional restrictions. However, using the variation in players' approximated wealth and empirical frequencies to end up in court for different types of victim-defendant matches, we can estimate preferences and non-monetary fighting abilities separately. The described strategy works well when we assume no selection bias at a pre-investigation stage. If, however, some matches (for example, $\left(w_D = \bar{w}^D,\, w_V = \underline{w}^V\right)$) are systematically underrepresented in our sample, the estimator must be adjusted respectively. Other identification restrictions will be discussed later.

Let γ and X denote the sets of parameters and controls, respectively.

Definition. $N_{\varepsilon \in [u_1,\, u_2]}\,(m,\, \sigma)$ denotes the truncation of a normally distributed random variable $\varepsilon \sim N\,(m,\, \sigma)$ for $\varepsilon \in [u_1,\, u_2]$. Parameters m and σ correspond to mean and standard deviation of the general normal distribution.[60]

Given the data available, we cannot distinguish x from p^h and work only with the expected punishment, xp^h. Assume xp^h is drawn from the following distribution:

$$xp^h \sim N_{xp^h \geq 0}\,(f\,(h,\, Z)\,,\, \sigma_x)$$

[60]The theoretical model we developed in Section 2.4 is well-defined only for non-negative values of xp^h, a, b, w_i, m_i, $i = \{V,\, D\}$. For this reason, one must restrict the supports of the underlying distributions.

where $f(h, Z)$ is a deterministic function of harm made (h) and other case- and region-specific controls (Z).[61] The draws of xp^h are independent across "victim-defendant" matches. For every case i, we impose the following restriction on the shape of $f(h, Z)$:

$$f(h_i, Z_i) = \lambda_0 + \lambda_1 h_i + \lambda_2 region_i$$

where

- h_i includes all characteristics of the accident, such as:
  1. A number of victims from different gender and age groups;
  2. Whether D and / or V were drunk;
  3. If D already has criminal and / or administrative records.

  All these controls enter the value of h_i linearly.
- $region_i$ contains dummy variables showing where the accident happened.

As it was mentioned before, all the cases have three possible outcomes:

1. $s = 1$: D and V settle among themselves (otherwise, $s = 0$)
2. $c = 1$: D and V do not settle, and the case goes to court (otherwise, $c = 0$):
   - $x_{obs} = 1$: the decision is known
   - $x_{obs} = 0$: the decision is not known
3. $nc = 1$: D and V do not settle, and the case does not go to court (for example, an investigator or a prosecutor can decide to close the file)

In practice, underlying parameters of the model, such as vindictiveness or non-monetary fighting abilities, depend on victim- and defendant-specific characteristics. For this reason, we impose following assumptions on the distributions of a, b, w_V, m_D, m_V, m_D:[62]

$$w_V^i \sim N_{w_V^i \geq 0}\left(\bar{w}_V^i, \sigma_V^w\right) \; where$$
$$\bar{w}_V = \alpha_0 + \alpha_1 SES_V^i + \alpha_2 gender_V^i + \alpha_3 child^i + \alpha_4 age_V^i + \alpha_5 \left(age_V^i\right)^2$$
$$w_D^i \sim N_{w_D^i \geq 0}\left(\bar{w}_D^i, \sigma_D^w\right) \; where$$
$$\bar{w}_D^i = \beta_0 + \beta_1 SES_D^i + \beta_2 gender_D^i + \beta_3 age_D^i + \beta_4 \left(age_D^i\right)^2 + \beta_5 edu_D^i + pcar_D^i$$
$$a^i \sim N_{a^i \geq 0}\left(\bar{a}^i, \bar{\sigma}_a\right) \; where$$
$$\bar{a}^i = \delta_0 + \delta_1 pedestrian_V^i + \delta_2 related^i + \delta_3 \bar{w}_V^i, \; \bar{\sigma}_a = \sqrt{\sigma_a^2 + \delta_2^2 Var\left(w_V^i\right)}$$

[61] The distribution of xp^h is the lower truncation of $N(0, \sigma_x)$ with the $xp^h \in [0, \infty)$ support.

[62] All draws are assumed to be independent across cases.

$$b^i \sim N_{b^i \geq 0}\left(\bar{b}^i,\, \bar{\sigma}_b\right)\ where$$
$$\bar{b}^i = \eta_0 + \eta_1 \bar{w}_D^i,\ \bar{\sigma}_b = \sqrt{\sigma_b^2 + \eta_1^2 Var\left(w_D^i\right)}$$
$$m_j^i \sim N_{m_j^i \geq 0}\left(\bar{m}_j^i,\, \sigma_j^m\right),\ j = \{V,\, D\}\ where$$
$$\bar{m}_j^i = \pi_0^j + \pi_1^j lawenf_j^i$$

where

- SES_j^i denotes socio-economic status of $l = \{V,\, D\}$;
- $child^i = 1$ if V's age is below 18;
- age_j^i indicates which age group $l = \{V,\, D\}$ belongs to;
- $pcar_D^i$ reflects a mean car price for D;
- $pedestrian_V^i = 1$ signals that V is a pedestrian;
- edu_D^i shows the highest degree D holds (in years of education);
- $related^i$ establishes if V knows D and how close their relationships are (for example, family members);
- $lawenf_l^i = 1$ specifies whether $l = \{V,\, D\}$ is a law enforcer or a government official.[63]

We comment on these assumptions briefly. To identify parameters that shape expected wealth ($\bar{w}_V$ and $\bar{w}_D$), we exploit the variation in players' socio-economic status (SES), gender, age, educational attainment, and imputed mean car prices (where applicable). Clearly, SES affects individual income and positively correlates with w_V and w_D. On average, we expect top-managers to display higher wealth than workers and people without a permanent job. Also, we take account of *gender* to explore the variation in resources available to males and females. Mean car prices ($pcar$) convey another piece of information about contestants' wealth. On average, those who drive more expensive vehicles are richer.[64] However, car prices become a relevant proxy for V's wealth if and only if the victim happened to be a driver or a passenger but not a pedestrian. As it was mentioned earlier, the information on V's car price is harder to get from the *fabulas* and may contain a lot of noise. For these two reasons, we do not include $pcar$ into w_V.

Child victims can display different wealth patterns because they do not have own income yet. To capture this source of variation in w_V, we assign a separate indicator, $child^i$, to the

[63]The *iid* assumption stems from random matching between victims and defendants in traffic accidents, which constitute unintentional crimes.

[64]Notice that $pcar$ enters w_D with the weight of 1. This allows us measure w_D in currency units and treat other coefficients as exchange rates between money and the controls of interest.

given group. Typically, individuals earn less in the beginning and in the end of their career, i.e. w_i, $i = \{V, D\}$ can display an inverse U-shape pattern with respect to age (keeping all other characteristics constant). For this reason, w_i, $i = \{V, D\}$ must be instrumented with age and age^2. Finally, well-paid jobs often require a college degree. Thus, better educational background can be associated with higher wealth.

Next, consider what drives contestants' preferences. In the beginning, we focus on the vindictiveness parameter a. To identify the value of interest, two sources of variation can be exploited. First, we look at victim-defendant relationship. Particularly, close relatives of the defendant who play on the victim's side may be less vindictive than strangers. Second, if richer individuals lose their ability to work and generate more income, they can extract higher vengeance benefits from D being punished. To capture this effect, w_V enters a. D's disutility of being punished also correlates with his wealth. For instance, those defendants who earn a lot or have a very promising career can lose more in case of punishment, especially if they get a real jail term.

In the end, we explain how m_V and m_D are shaped. As it was emphasized before, being a law enforcer or a government official gives two advantages. First, it allows the individual learn the system (its institutional setting, legal procedures etc.) better and act faster if the accident happens. Second, those who work in the given sectors operate in the network of law enforcers, build new connections and can use this influence if needed. Both aforementioned assets are clearly non-monetary and affect the contest outcome.[65] Hence, we use $lawenf$ to explain m_V and m_D .

Generally, m_V and m_D cannot be separately identified (see ***Result 3*** and the discussion on Page 142). Nevertheless, instrumenting m_V and m_D with $lawenf$ makes it possible to quantify the advantage (or disadvantage) law enforcers and government officials have in non-monetary fighting abilities. In the data, we observe four types of matches:

1. Both V and D do not work in law enforcement or in the government, i.e. $lawenf_V = lawenf_D = 0$;

2. Only V (D) works in law enforcement or in the government, i.e. $lawenf_V = 1$, $lawenf_D = 0$ ($lawenf_V = 0$, $lawenf_D = 1$);

3. Both V and D work in law enforcement or in the government, i.e. $lawenf_V = lawenf_D = 1$.

Let $\left(\frac{m_V}{m_D}\right)_{\{lawenf_V, lawenf_D\}}$ denote the ratio of non-monetary fighting abilities for each $\{lawenf_V, lawenf_D\}$ match. Estimating the model, we obtain four values:

$$\widehat{\left(\frac{m_V}{m_D}\right)}_{\{0,0\}} = r_1, \; \widehat{\left(\frac{m_V}{m_D}\right)}_{\{1,0\}} = r_2$$

[65] See Subsection 2.5.2 for more details.

$$\widehat{\left(\frac{m_V}{m_D}\right)}_{\{0,1\}} = r_3, \; \widehat{\left(\frac{m_V}{m_D}\right)}_{\{1,1\}} = r_4$$

Using this information, one can compute a lower bound for the relative difference in non-monetary fighting abilities victims and defendants from different groups display:

$$\widehat{\frac{m_V^1}{m_V^0}} = \min\left\{\frac{r_4}{r_3}, \frac{r_2}{r_1}\right\}, \; \widehat{\frac{m_D^1}{m_D^0}} = \min\left\{\frac{r_1}{r_2}, \frac{r_2}{r_4}\right\}$$

where m_i^0 (m_i^1) corresponds to the m_i value under $lawenf_i = 0$ ($lawenf_i = 1$), $i = \{V,\, D\}$.

The random nature of traffic offenses and the modeling assumptions imposed on key parameters allow us identify the variance of underlying noise distributions. The expected punishment xp^h must not affect players' preferences and fighting abilities. Including region-specific controls and distinguishing between different types of victims (children, females etc.) help us isolate common shocks judges face when making their decisions about sanctions.[66] Hence, the unexplained variation can be associated with a noise term.

Players' wealth levels, w_V and w_D, are assumed to be uncorrelated with their preferences and non-monetary fighting abilities. Moreover, given that defendants and victims match at random, w_V and w_D must be independently drawn from different distributions. Here, we assume that two sides of the conflict can display contrasting wealth patterns. The identification becomes possible when we use "Car vs. car" accidents as a control group for "Car vs. pedestrian" offenses. On average, those who happen to own a car can be richer than individuals without vehicles. Hence, w_V and w_D are driven by different data generating processes. Controlling for systematic patterns (socio-economic status, age, educational attainment, imputed mean car prices), we attribute the residual variation to wealth shocks and identify σ_V^w and σ_D^w.

As ***Result 3*** shows, contestants' preferences can be separately identified, and we model them as a function of individual wealth. For the vindictiveness parameter a, there is one more instrumental variable – victim-defendant relationship – that helps in capturing other systematic patterns.[67] Again, the randomness of traffic accidents allows us assume a zero correlation between a and b, which is especially true when one concentrates on "Car vs. pedestrian" matches. Hence, all unexplained variation in players' preferences is treated as shocks to the corresponding variables. Similar arguments apply to the identification of σ_V^m (σ_D^m) where m_V (m_D) is assumed to be independent of wealth, preference parameters, case- and opponent-specific characteristics.

Now, we employ the theoretical model to construct the likelihood function. Let $F_{G \subseteq R^7_{\geq}}(g)$, $G = \left\{w_V,\, w_D,\, a,\, b,\, m_V,\, m_D,\, xp^h\right\}$ denote a joint distribution of w_V, w_D, a, b, m_V, m_D, and xp^h.

[66] Ideally, one should also control for policy department and court-specific fixed effects. Unfortunately, this information is not available for all cases.

[67] Also, it is crucial that the harm made does not shape preferences for revenge (a).

Assume I_j, $j = \{1, ..., 4\}$ is equal to a unity if equilibrium j is played (otherwise, $I_j = 0$).[68] Then, a complete data likelihood for every case i (L_i^c) looks as follows:

$$L_i^c(\gamma, X_i) = \sum_{j=1}^{4} I_{ij} \left(\left(I_{ij}^S\right)^{s_i} \left[\left(1 - I_{ij}^S\right) \left(P_C^{ij} \left(xp^h\right)_i^{x_{obs}^i} \right)^{c_i} \left(1 - P_C^{ij}\right)^{nc_i} \right]^{1-s_i} \right)$$

where

- $I_{ij}^S = 1$ if D and V prefer to settle in equilibrium j for given parameter values and budget constraints;
- $\left(xp^h\right)_i$ represents the expected in-court punishment.

Since we do not observe w_V, w_D, a, b, m_V, m_D, and xp^h directly, I_{ij} and I_{ij}^S must be replaced with corresponding probabilities. Also, one has to take expected values of $\left(xp^h\right)_i$ and P_C^{ij} (a probability to end up in court for case i).

To give an example, consider equilibrium 1 where budget constraints of both D and V are non-binding. This outcome emerges if and only if:[69]

$$\begin{cases} w_V \geq m_V \frac{xp^h a^2 b m_D}{(am_D + bm_V)^2} \\ w_D \geq m_D \frac{xp^h b^2 a m_V}{(am_D + bm_V)^2} \end{cases}$$

Then, we can define a probability to observe equilibrium 1 in the data:

$$P_1 = Pr\left(w_V \geq m_V \frac{xp^h a^2 b m_D}{(am_D + bm_V)^2},\ w_D \geq m_D \frac{xp^h b^2 a m_V}{(am_D + bm_V)^2} \right)$$

where P_1 can be compute given $F_{G \subseteq R^7_{\geq}}(g)$ and the assumptions on the distributions of w_V, w_D, a, b, m_V, m_D, and xp^h.[70] Similarly, one can find the probabilities to observe equilibria 2, 3, and 4:

$$P_2 = Pr\left(w_V \geq m_V \frac{xp^h a^2 b m_D}{(am_D + bm_V)^2},\ w_D < m_D \frac{xp^h b^2 a m_V}{(am_D + bm_V)^2} \right)$$
$$Pr\left(w_V < m_V \frac{xp^h a^2 b m_D}{(am_D + bm_V)^2},\ w_D \geq m_D \frac{xp^h b^2 a m_V}{(am_D + bm_V)^2} \right)$$
$$P_4 = Pr\left(w_V < m_V \frac{xp^h a^2 b m_D}{(am_D + bm_V)^2},\ w_D < m_D \frac{xp^h b^2 a m_V}{(am_D + bm_V)^2} \right)$$

Now, for every equilibrium outcome j, we can find a probability to settle, P_j^s. Recall conditions (2.4.1) and (2.4.2) where the former indicates D's willingness to settle and the latter states when it is feasible to prevent the fight. Then, a probability to settle becomes

[68]The equilibrium type depends on whose budget constrain binds in the optimum. See the proof of Proposition 2.1 for more details.

[69]See the proof of Proposition 2.1 for more details.

[70]There is no analytical solution for P_1. To approximate this value, we simulate the $F_{G \subseteq R^7_{\geq}}(g)$ distribution for given parameters and recover P_1.

$$P_j^s = Pr\left(\pi_{D,j}^* \leq -\pi_{V,j}^*,\, \pi_{V,j}^* \leq w_D \,|\, j\right)$$

where $\pi_{D,j}^*$ and $\pi_{V,j}^*$ reflect contestants' payoffs in equilibrium j.[71]

Further, we turn our attention to scenario-specific probabilities to end up in court (P_C^j). In the complete data case, these values look as follows:

$$P_C^1 = \frac{am_D}{am_D + bm_V},\, P_C^2 = 1 - \sqrt{\frac{w_D m_V}{m_D axp^h}}$$
$$P_C^3 = \sqrt{\frac{w_V m_D}{m_V bxp^h}},\, P_C^4 = \frac{w_V m_D}{w_V m_D + w_D m_V}$$

Since we only know the distributions of w_V, w_D, a, b, m_V, m_D, and xp^h, one needs to take expected values of P_C^j, $j = \{1, ..., 4\}$:

$$\bar{P}_C^j = E\left(P_C^j \,|\, j\right)$$

P_C^j, $j = \{1, ..., 4\}$ are well-defined over supports of the corresponding conditional distributions.[72] The probability to avoid the court stage is just a complement of P_C (or its expected value).

For each case that goes to court, the decision is either available ($x_{obs} = 1$) or not ($x_{obs} = 0$). In the data, x_{obs} includes various types of punishment ranging from different limitations of freedom to real prison sentences. To rank these options in utility (or disutility) terms, one might propose a scale. However, it requires extensive robustness checks. Instead, we employ the following (although simplistic) assumption:

$$x^* = \begin{cases} 1\, when\, D\, gets\, a\, real\, jail\, term \\ 0,\, otherwise \end{cases}$$

where x^* denotes the punishment observed. With this formulation, we arrive to a typical binary choice model:

$$x^* = \begin{cases} 1\, if\, xp^h \geq t \\ 0\, if\, xp^h < t \end{cases}$$

and t indicates a threshold value to be estimated. Then, the probability to observe a particular decision becomes:

$$P_{x^*=1} = 1 - Pr\left(xp^h < t\right),\, P_{x^*=0} = Pr\left(xp^h < t\right)$$

[71] See the proof of Proposition 2.1 for closed-form expressions of $\pi_{D,j}^*$ and $\pi_{V,j}^*$.

[72] All values under square roots stay positive; $P_C^j \in [0, 1]$.

With all computations provided, an expected likelihood function for case i (L_i^e) becomes:

$$L_i^e(\gamma,\, X_i) = \sum_{j=1}^{4} P_{ij} \left(\left(P_{ij}^S\right)^{s_i} \left[\left(1 - P_{ij}^S\right) \left(\bar{P}_C^{ij} \left[\left(P_{x^*=1}\right)^{x_i^*} \left(P_{x^*=0}\right)^{1-x_i^*} \right]^{x_{obs}^i} \right)^{c_i} \left(1 - \bar{P}_C^{ij}\right)^{nc_i} \right]^{1-s_i} \right)$$

Since draws of w_V, w_D, a, b, m_V, m_D, and xp^h are independent over cases, a full-sample expected likelihood (L^e) is just a product of L_i^e:

$$L^e(\gamma,\, X) = \prod_{i=1}^{N} L_i^e(\gamma,\, X_i)$$

where N is a sample size. Taking logarithms, we get:

$$l^e(\gamma,\, X) = log\left(L^e(\gamma,\, X)\right)$$

Finally, to have a well-defined game, all underlying parameters of the model (such as preferences, monetary and non-monetary fighting abilities) must be non-negative. The resulting program to solve is:

$$\max\nolimits_\gamma \left\{l^e(\gamma,\, X)\right\}$$
$$s.t.\ \min\nolimits_i \left\{\bar{w}_D^i,\, \bar{w}_V^i,\, \bar{a}^i,\, \bar{b}^i,\, \bar{e}^i,\, \bar{m}^i,\, \left(\bar{xp}^h\right)_i\right\} \geq 0$$

Since our econometric model has a parameter-dependent support, we use a derivative-free numerical algorithm to solve the optimization program. Typically, the estimators of this class are consistent, but not necessarily asymptotically normal. For this reason, we do not approximate the variance-covariance matrix as the inverse of Hessian (this procedure relies on the asymptotic normality assumption), but compute bootstrap standard errors.[73]

We develop the following estimation procedure. To capture the difference in non-monetary fighting abilities, one should concentrate on "car vs. pedestrian" cases (see Subsection 2.5.2 for the reduced form evidence). However, working with accidental "victim-defendant" combinations only, we lose the information on non-random matches, where V and D know each other. As a result, V's vindictiveness (a) cannot be identified. For this reason, we run the estimation on two different samples, N_{all} and N_p.

The first dataset, N_{all}, includes all types of criminal traffic offenses for 3 regions: Sverdlovsk Oblast (732 cases), Chelyabinsk Oblast (868 cases), and Permsky Krai (479 cases).[74] These regions are located in the same geographic area (namely, Ural) and share identical climate conditions.

[73]Hirano and Porter (2003) show the maximum likelihood estimator can be asymptotically inefficient in models with parameter-dependent support. It results in bigger standard errors, but does not affect unbiasedness and consistency properties. For this reason, we still find the present MLE estimator sufficient; however, the procedure must be adjusted in the future.

[74]Other regions will be added later.

Moreover, they display comparable socio-economic characteristics. Overall, we expect that drivers in the given regions should have analogous behavioral patterns.

Another subsample, N_p, deals with "car vs. pedestrian" accidents only. It includes 9 regions and 1055 cases in total. Also, the N_p sample has a sufficient number of law enforcers and government officials on both sides of the process. With this dataset, we abstract from non-random matches and aim to estimate non-monetary fighting abilities of the parties (m_V and m_D).[75]

Region	Number of cases	Number of victims who are law enforcers or government officials	Number of defendants who are law enforcers or government officials
Moscow	123	0	3
Moscow Oblast	126	4	2
Permski Krai	122	2	1
Sverdlovsk Oblast	124	3	1
Tjumen Oblast	111	2	0
Omsk Oblast	74	1	0
Novosibirsk Oblast	123	2	1
Altayski Krai	123	4	3
Chabarovsk Krai	128	7	1
Total	1055	25	12

2.5.4 Estimation Results

Table 2.4 (Table 2.10) reports estimation results for the sample of all criminal traffic offenses ("car vs. pedestrian" matches). For both N_{all} and N_p, the victims tend to have lower expected wealth than the defendants, and the degree of resource imbalances (θ) is 1.7 times bigger in the N_p case:[76]

$$N_{all}:\ mean\left(\bar{w}_D^i\right) = 55'806.6,\ mean\left(\bar{w}_V^i\right) = 6'056.7$$

$$\theta_{all} = mean\left(\frac{\bar{w}_D^i - \bar{w}_V^i}{\max_i\left\{\bar{w}_D^i - \bar{w}_V^i\right\}}\right) = .3$$

$$N_p:\ mean\left(\bar{w}_D^i\right) = 2'257.3,\ mean\left(\bar{w}_V^i\right) = 903.9$$

$$\theta_p = mean\left(\frac{\bar{w}_D^i - \bar{w}_V^i}{\max_i\left\{\bar{w}_D^i - \bar{w}_V^i\right\}}\right) = .52$$

In the N_{all} case, w_V and w_D increase in $age \geq 18$, i.e. on average, elder individuals are reacher.[77] For "car vs. pedestrian" matches (the N_p dataset), the pattern differs. Specifically, we find the evidence of the inverse U-shape relationship between age_j, $j = \{V, D\}$ and individual wealth (keeping other characteristics constant):

[75] Later on, the two identification approaches must be incorporated.

[76] The distribution of $\bar{w}_V^i$ has more mass to the left of its mean (Figure 2.6.1 for N_{all} and Figure 2.6.3 for N_p). The distribution of $\bar{w}_D^i$ is left (right) skewed in the N_{all} (N_p) case (Figure 2.6.2 and Figure 2.6.4, respectively).

[77] Individuals whose age is under 18 enter the $child_V = 1$ group.

$$argmax_{age_V}\{\bar{w}_V\} \approx 43,\ argmax_{age_D}\{\bar{w}_D\} \approx 35$$

where age_V and age_D are measured in years. In words, V's (D's) wealth achieves its maximum when the individual turns 43 (35). The difference arises from the discrepancy in V's and D's age composition. Actually, 70% of the defendants are younger than 40, and 60% of the victims turn to be older than 30 (see Table 2.2).

Next, consider how the victim's socio-economic status (SES) affects her wealth. In case of the N_{all} sample, most of the patterns are quite predictable. For example, workers, welfare recipients, and retired individuals tend to be poorer than office employees (keeping other characteristics constant). At the same time, victims who happen to be top-managers or budget office workers display the lowest wealth level. The former observation goes against the intuition, although the corresponding coefficient is not statistically significant. Also, the positive impact age has on w_V can offset the negative effect of top-managers' SES: on average, these individuals are elder than their peers.[78] Once we concentrate on random matches (the N_p sample), top-managers display higher wealth than the control group (unemployed individuals). Nevertheless, the magnitude of the effect is still smaller than for office workers, students or even retired individuals. The indicated observation can be justified with a small number of top-manager victims in the sample (less than 1%, see Table 2.2). In fact, these individuals are less likely to be pedestrians and also tend to drive expensive cars, which protect them and their passengers from serious injuries. Hence, those top-manager victims who have entered the sample might not significantly differ from other socio-economic groups.

D's wealth behaves more predictably with respect to his socio-economic status. When we focus on all criminal traffic offenses, top-managers display higher values of w_D than their peers, excluding workers. Overall, the frequency of observing this socio-economic group on the defendants' side is higher: top-managers constitute more than 1% of the offenders' population (see Table 2.2). This explains why the effect of their SES on w_D turns to be more intuitive than in case of w_V. Students and other individuals are the poorest offenders in the sample. For the N_p dataset, every SES unit performs better than the baseline (namely, unemployed individuals). The top-manager defendants still hold more wealth, but the magnitude of the effect becomes smaller than in the N_{all} case.

Child victims ($age_V < 18$), whose resources depend on their parents' socio-economic status, tend to have less wealth than adults ($age_V \geq 18$). Although the coefficient in front of $child_V$ is positive, its effect on w_V turns to be weaker than the impact age_V has on V's wealth. This result holds for both samples. Remarkably, the effect of gender on w_V and w_D is positive: on average, female victims and defendants have more resources (keeping other characteristics constant). Also, the impact of $gender_V = 1$ on w_V becomes weaker in the N_p case. One can assume that women

[78] In the N_{all} sample, the average age of top-manager victims reaches 36.4. This value exceeds its counterpart for most of the SES groups (except entrepreneurs, retired and unclassified individuals).

Table 2.4: Estimation Results: All Cases (N_{all})

Variable	Coefficient	St. Error
Victim's Wealth		
Intercept	126.1***	36.505
SES_V:		
worker	75.67***	7.272
office worker	101.95***	7.326
top-manager	−1.13	29.254
entrepreneur	87.94***	13.368
budget office worker	−4.96	9.346
student	229.19***	20.938
welfare recipient	90.7**	41.585
retired	80.56**	31.871
other	8.43	11.885
$gender_V$	49.93**	23.961
$child_V$	245.91***	4.104
age_V	−59.33**	28.231
age_V^2	4.68	10.379
Defendant's Wealth		
Intercept	2.46	4.175
SES_D:		
worker	101.2***	6.091
office worker	58.34***	8.566
top-manager	97.89**	40.272
entrepreneur	42.54**	20.718
budget office worker	76.14***	1.221
student	−1.81	51.428
welfare recipient	47.4***	11.12
retired	54.77***	5.511
other	−13.17	38.651
$gender_D$	26.04	42.122
age_D	110.92***	2.663
age_D^2	38.03**	17.899
edu_D	55.76**	23.568
$pcar_D$	1	None
Vindictiveness (a)		
Intercept	74.87***	6.893
pedestrian	98.43***	.295
related:		
acquaintance	9.12	36.402
cohabitant	11.77	21.805
relative	15.2	29.053
close relative	10.24***	3.078
w_V	1.49	3.491
Defendant's Disutility of Punishment (b)		
Intercept	22.75*	12.937
w_D	79.92**	36.435

Variable	Coefficient	St.Error
Non-Monetary Fighting Abilities		
Victim:		
Intercept	19.04	25.808
$lawenf_V$	19.77	47.113
Defendant:		
Intercept	8.23	39.166
$lawenf_D$	23.63	19.48
Accident and Harm		
Number of		
dead victims	156.26***	25.078
victims with serious injuries	71.63***	5.004
dead minor victims	82.58*	44.233
minor victims with serious injuries	95.48***	9.647
dead female victims	8.11**	3.247
female victims with serious injuries	−17.7	12.149
dead minor female victims	82.24***	2.021
minor female victims with serious injuries	84.06***	27.923
pedestrian	2.83	22.32
$drunk_D$	38.5	29.559
$drunk_V$	19.22***	1.444
$crimehist_D$	65.72***	9.35
$admhist_D$	89.65**	36.907
$record_D$	13.7	10.002
Region-specific controls	Yes	
t	325.76***	22.834
Underlying Distributions		
σ_V^w	54.42	39.528
σ_D^w	29.99***	6.808
σ_a	73.24**	28.696
σ_b	16.18	32.342
σ_V^m	26.35*	15.471
σ_D^m	120.23***	20.114
σ_x	225.03***	51.537
$log(L)$	−3497.16	
N	2079	

who happen to drive a car are richer than their male peers.[79] This may be especially true in Russia where the culture of female drivers has been developing recently.[80] The fact that this gender group forms only 9% of the defendants' population speaks in favor of the latter argument (see Table 2.2).[81] Finally, D's who spent more years in education show higher wealth: the effect of edu_D is positive on both samples and even more pronounced in the N_p case.

Next, we analyze what defines V's expected vindictiveness ($\bar{a}$) and D's disutility of punishment ($\bar{b}$). As we hypothesized before, both a and b increase in individual wealth: the coefficients in front of w_V and w_D are positive, although not always statistically significant. With these effects, we observe ($a < b$) with probability 1 in all the matches. Specifically, D's disutility of punishment exceeds V's vengeance benefit. At the same time, the N_{all} sample features smaller heterogeneity in a and b:

$$N_{all} : mean\left(\frac{\left|\bar{a}^i - \bar{b}^i\right|}{\max_i \left\{\left|\bar{a}^i - \bar{b}^i\right|\right\}}\right) = .3$$

$$N_p : mean\left(\frac{\left|\bar{a}^i - \bar{b}^i\right|}{\max_i \left\{\left|\bar{a}^i - \bar{b}^i\right|\right\}}\right) = .8$$

The finding is driven by the fact that victims tend to have less wealth than their opponents, and this shifts the distribution of $\bar{a}$ to the left of $\bar{b}$. Moreover, the resource inequality becomes more pronounced in the N_p case where cars randomly match with pedestrians. Going back to the analysis of Section 2.4, for the given distributions of w_V and w_D, the defendants always prefer to settle when they have enough money to make the optimal offer S.

Given the estimation results for the N_{all} sample, the relation of V to D indeed shapes V's preferences. In fact, strangers (pedestrians) tend to be 9.6 times more vindictive than close relatives of the defendants. Excluding w_V and w_D from the specifications of $\bar{a}$ and $\bar{b}$ due to their insignificance, we observe ($\bar{a} > \bar{b}$) for the two datasets:

$$N_{all} : \min(\bar{a}) = 74.87 > 22.75 = \bar{b}$$

$$N_p : \bar{a} = 89.71 > 73.21$$

With probability .92, victims have strong preferences for revenge, and settling with offenders becomes inefficient. In this case, feasible but not desirable agreements can emerge (see Proposition 2.4).[82]

[79] According to surveys conducted in Russia in 2012, a typical female driver was a top-manager or a young mother with average or above average income. See https://www.dp.ru/a/2012/03/27/CHislo_zhenshhin_za_rulem_v_R.

[80] In 2012, females constituted 24% of Russian drivers, which was 1.7 times more than in 2007. See https://www.dp.ru/a/2012/03/27/CHislo_zhenshhin_za_rulem_v_R.

[81] An alternative explanation of the observed defendants' gender composition might be the difference in risk preferences for males and females.

[82] Based on simulations, this scenario is not very frequent. For the N_p sample, feasible but not desirable settlements appear with probability 1.3e-4 (1e-4) in equilibrium 1 (3).

For both samples, we do not find the evidence that law enforcers and government officials have better bargaining positions. This can be explained as follows. First, since law enforcers and government officials constitute a relatively small subgroup (around 1% of victims and 2% of defendants), one should increase the sample size to capture any systematic patterns these individuals display. Actually, the reduced form analysis (Subsection 2.5.2) uses all the information available and signals in favor of this argument. Another, and potentially more problematic, concern is non-randomness of the sample. The group of law enforcers and government officials is heterogeneous enough: it includes high-rank individuals, as well as regular employees who perform minor tasks. The former cohort can use their non-monetary assets (namely, connections and influence) in order to avoid the investigation stage. As a result, these individuals do not enter the sample. The original dataset includes around 14'000 records where offenders are missing. It might be that individuals who managed to close their cases before the investigation had started enter the given subset. Hence, those law enforcers and government officials whom we observe may not differ from other defendants much.[83] Similar arguments apply to top-managers who occupy very high positions. Thus, one must find a way to account for this potential selection bias and adjust the estimator

Finally, we comment on main determinants of the expected punishment. Not surprisingly, the probability to get a real sentence increases with the number of dead / seriously injured victims, and the former contributes the most. The effect of killing or hurting a child on the expected punishment is positive and significantly higher compared to the case where a female victim dies. At the same time, the probability to end up in prison decreases if the accident causes only serious bodily injuries for a woman. Hitting a pedestrian, being drunk, and having criminal or administrative records enlarge the harm and lead to a stricter punishment.

For the N_{all} sample, matching with a drunk victim increases the probability to get a real sentence as well. Given that we consider all criminal traffic offenses, this result has the following interpretation. If the victim who was driving a car happened to be drunk or intoxicated, the accident is likely to have severe consequences. As a result, the expected punishment for the offender may rise. Also, here we allow for non-random matching between victims and defendants. Then, it becomes probable for drunk drivers to have intoxicated passengers who get hurt if the accident happens. Hence, observing $drunk_V = 1$ positively correlates with facing $drunk_D = 1$, which increases the expected punishment. In case of the N_p sample, $drunk_V = 1$ reduced D's probability to end up in prison. Here, being injured might refer to V's own fault and mitigate D's guilt. Thus, focusing on random matches, we isolate a positive correlation between $drunk_V = 1$ and $drunk_D = 1$.

[83] Actually, they can even be in a disadvantaged position and display the behavior similar to low-status individuals.

Further, we run the goodness-of-fit tests and check whether the model replicates key observed patterns well. Specifically, the following empirical frequencies are matched against their simulated counterparts:

1. $E(P_S)$ – a probability to observe the settlement decision;
2. $E(P_C \,|\, no\, S)$ – a probability that the case ends up in court given no settlement has happened;
3. $E(P(x^* = 1) \,|\, C)$ – a probability that the defendant gets a real sentence if the case has reached the court stage.

Table 2.5 reports the statistics for all cases in the N_{all} sample. The model replicates main stylized facts well. Also, the Pearson's χ^2 goodness-of-fit test cannot reject the null hypothesis that the simulated distribution coincides with its empirical counterpart. Then, we divide the dataset into groups based on victim- and defendant-specific characteristics. In particular, we compute empirical and simulated moments for individuals from different age and socio-economic cohorts. On top of this, we trace the defendants who have a criminal history.

Group-specific moments are summarized in Table 2.9. Generally, the model gets very close to the empirical frequencies. Also, we do not detect any tendency towards systematic over- or underestimation of the moments. The worst performance refers to the group of child victims. Here, the model predicts significantly higher probabilities to settle and face a real sentence than observed in the data.[84] With regard to defendant-specific characteristics, we underestimate (overestimate) $E(P_S)$ ($E(P(x^* = 1) \,|\, C)$) for female offenders and college graduates. With all other groups, the model performs quite well.

We repeat the same analysis for the "car vs. pedestrian" sample (Table 2.11). On average, the model performs worse than in the N_{all} case. It replicates the probability to face a real sentence quite well. However, the model systematically overestimates (underestimates) $E(P_C \,|\, no\, S)$ ($E(P_S)$), although the bias is not very big. At the same time, for some groups, such as child victims or defendants with a criminal record, the model performs better than its N_{all} counterpart. Overall, increasing the sample size and controlling for a possible selection bias discussed earlier must improve the predicting power of the model.

Finally, we use the estimates to evaluate the cost of bargaining (for settled lawsuits) and fighting (for non-settled cases) the defendants face. As it was mentioned before, the victims do not display high vengeance benefits when w_V and w_D enter the specifications of $\bar{a}$ and $\bar{b}$ ($a < b$ with probability 1). Then, the offenders always prefer to settle if they have enough resources to pay the amount of S (see Section 2.4). To make players' payoffs more comparable, we weight the disutility D

[84]This pattern disappears once we concentrate on "car vs. pedestrian" matches.

Table 2.5: Goodness-of-Fit: All Cases (N_{all})

Moments	$E(P_S)$		$E(P_C \mid no\, S)$		$E(P(x^* = 1) \mid C)$	
Samples	Data (1)	Sim. (2)	Data (3)	Sim. (4)	Data (5)	Sim. (6)
All cases:	.168	.160 (1.9e-4)	.451	.435 (3.4e-4)	.406	.444 (5.1e-4)
Pearson's χ^2 stat.	8.89					
Critical χ^2_3 ($\alpha = .99$)	9.21					

Note:
To simulate the model, 1'000 draws from the estimated distributions of w_V, w_D, a, b, m_V, m_D, and xp^h are taken. This procedure is repeated 100 times.
$E(P_S)$ denotes an expected probability to settle. $E(P_C \mid no\, S)$ reflects an expected probability to end up in court given no settlement. $E(P(x = 1 \mid C))$ defines an expected probability to get a real sentence once the case goes to court. For the data, $E(P_S)$, $E(P_C \mid no\, S)$, and $E(P(x = 1 \mid C))$ correspond to a frequency of observing $s = 1$, $c = 1$, and $x^* = 1$, respectively. In case of simulations, the values are computed based on the estimated distributions. Standard deviations are reported in parentheses.

encounters by his monetary fighting ability, w_D. Let p_w^s and p_w^c denote D's relative payoff when he settles with the victim and ends up in court, respectively:

$$\begin{aligned} p_w^s &= -\tfrac{S}{w_D} \\ p_w^c &= -\tfrac{\pi_D^*}{w_D} \end{aligned}$$

where π_D^* defines D's equilibrium payoff. Table 2.6 compares the average values of p_w^s and p_w^c for the two sample. If the conflicting parties settle, the defendant pays a much lower cost than in the alternative scenario (–.007 against –810.78 for all cases in the N_p sample). The effect does not vanish even when we focus on different types of crime (1 dead pedestrian vs. 1 pedestrian with serious bodily injures). One can also interpret p_w^s and p_w^c as a punishment the offender faces. Hence, those defendants who did not manage to settle must suffer significantly more than their peers who committed similar crimes but had better bargaining positions. On top of resource imbalances, this induces the inequality before the law, which may constitute an important concern for the society. We discuss this aspect later.

Policy Experiments and the Discussion

Now, we run counterfactual experiments with the estimates obtained for the two samples. The first issue to address is how the ban of settlements would affect the prison population. To answer this

Table 2.6: Expected Relative Payoffs for Defendants in Settled and Non-Settled Cases

Payoff	N_{all} **Sample**	N_p **Sample**
Expected relative payoff for settled cases ($-S/w_D$)	–.008 (4.1e-5)	–.007 (8.9e-5)
with 1 dead pedestrian	–.007 (1.1e-4)	–.003 (1e-4)
with 1 seriously injured pedestrian	–.009 (5.2e-5)	–.008 (8.8e-5)
Expected relative payoff for non-settled cases ($-\pi_D^*/w_D$)	–2903.89 (1.327)	–810.78 (.554)
with 1 dead pedestrian	–3107.27 (3.569)	–1'102.95 (.956)
with 1 seriously injured pedestrian	–2610.12 (1.958)	–657.54 (.783)

Note:
To simulate the model, 1'000 draws from the estimated distributions of w_V, w_D, a, b, m_V, m_D, and xp^h are taken. This procedure is repeated 100 times. Standard errors are reported in parentheses.

question, we eliminate the settlement stage and assume the game starts from the contest. With probability P_C, the case ends up in court, and the defendant faces the expected punishment xp^h. According to our assumptions, xp^h turn to be a real sentence ($x^* = 1$) if and only if the realization of xp^h exceeds the threshold t:

$$x^* = 1 \Leftrightarrow xp^h \geq t$$

To identify imprisoned defendants with the two sources of uncertainty (namely, P_C and $P\left(x^* = 1 \,|\, C\right)$), we apply the following rule:

$$prison = 1 \Leftrightarrow P_C P\left(x^* = 1 \,|\, C\right) \geq \bar{\rho}$$

where $\bar{\rho} \in (0, 1)$ is a threshold value that can be non-parametrically inferred from the data. In the beginning, we simulate the model for the N_{all} sample (all criminal traffic offenses). If pre-court agreements were forbidden, on average, 30% of the previously settled lawsuits (106 observations) would close with the defendants being imprisoned.[85] This can raise the cost of the society for two reasons. First, solving all cases in court puts an additional pressure on prosecutors and judges who have limited resources. Second, increasing the incarceration rate forces the society to redirect

[85] In the data, 353 cases (16.84% of all observations) settled.

more money to the prison system. For instance, Russia paid €2.2 per day for one incarcerated person in 2012. The total budget of the country's prison system reached €5.4 billion.[86] With these numbers, the monetary cost of keeping 106 additional individuals in prison for one year would amount to €85'118.[87] If we extrapolate this result to the full dataset (56'000 cases), the ban of "victim-defendant" settlements could increase the prison population by 2'856 inmates and cost Russia €2.3 million per year.

Next, consider the case of randomly matched victims and defendants (the N_p sample). In the data, 172 disputes (16.3% of all observations) are solves out-of-court. The ban of settlements leads to 69 more defendants going to prison.[88] If all 69 individuals get a 1-year incarceration sentence, the monetary cost of increasing the prison population reaches €55'407.[89]

Another question to investigate is how enlarging D's resources (his monetary fighting abilities) influences the case outcome. The effect is two-fold. More resources available allow the defendant make better offers and settle with stronger victims ("volume effect"). Also, higher values of w_D drive the amount of S down ("price effect").[90] Overall, a pre-court case resolution becomes easier when D's monetary fighting abilities improve.

Table 2.7 (2.12) provides simulation results for the N_{all} (N_p) sample. Here, all case-specific characteristics, except w_D, are kept the same. Particularly, a defendant with the given budget w_D is matched against the universe of victims from N_{all} (N_p). As one can see, relaxing D's resource constraint indeed allows the defendant to settle more often: $E(P_S)$ steadily increases with w_D. The average offer, however, tends to display an inverse U-shape: in the beginning, it grows with w_D, reaches the maximum at $w_D = 8\acute{}443\acute{}750$ ($w_D = 41\acute{}653$ in the N_p case) and then starts decreasing.[91] This pattern has the following explanation. When D has limited resources and his budget constraint relaxes slightly, he can afford much better offers and improve the settlement probability significantly. Here, the "volume effect" dominates the "price effect", and the average settlement offer rises.[92] If D holds a sufficient amount of resources, he is already able to reach an agreement with many victim types. For this reason, improving D's monetary fighting ability does not result in a pronounced "volume effect". However, it triggers the "price effect" because now the defendant can push the optimal settlement offer down (see Proposition 2.3 and the discussion on page 129 for more details). Thus, the average amount of S declines.

In Section 2.4, we showed that V and D can fail to achieve a settlement agreement even if

[86]The corresponding expenditures for France amounted to €98 per day and €2.4 billion, respectively. See http://www.rbc.ru/society/11/02/2015/54db24779a794752506f1ebf.

[87]The cost is even higher if the sentence exceeds one year.

[88]40% of the previously settled cases end up with a real prison term.

[89]The calculation is based on €2.2 per day for one incarcerated person.

[90]See Proposition 2.3 and the discussion on page 129 for more details.

[91]The pattern is more pronounced in the N_p case.

[92]Settling with mighty victims requires higher offers.

Table 2.7: The Effects of Increasing D's Wealth: All Cases (N_{all})

Moments	$E(P_S)$	N_S	$\bar{S}$
Wealth			
$w_D^1 = 13'099$	5.8e-3	12	20.6
	(2.3e-5)	(.05)	(.13)
$w_D^2 = 261'991$	.1	200	7'187.9
	(5.7e-5)	(.12)	(10.45)
$w_D^3 = 654'979$	.15	313	21'883.7
	(6.3e-5)	(.13)	(28.59)
$w_D^4 = 2'043'305$	.21	445	67'156.5
	(5.3e-5)	(.11)	(74.08)
$w_D^5 = 3'377'500$	.23	485	91'844.8
	(4.7e-5)	(.1)	(120.89)
$w_D^6 = 5'066'250$	.24	506	105'570.7
	(2.3e-5)	(.05)	(115.54)
$w_D^7 = 6'755'000$	.247	515	**107'443**
	(1.6e-5)	(.03)	(162.87)
$w_D^8 = 8'443'750$	.25	518	102'412.9
	(9.2e-5)	(.02)	(156.02)

Note:
To simulate the model, 1'000 draws from the estimated distributions of w_V, w_D, a, b, m_V, m_D, and xp^h are taken. This procedure is repeated 100 times. w_D^i, $i = \{1, ..., 8\}$ correspond to rescaled and sorted 0, .5, and 1 quantiles of $\bar{w}_D^i$'s estimated distribution. $\bar{w}_D^i$ is measured in rubles. Standard errors are reported in parentheses.

the defendant has enough resources to make the optimal offer S. In other words, D finds it more attractive to enter the contest stage because the amount of S is sufficiently high. Formally, this requires

$$\begin{cases} \pi_D^* > -S \\ S \leq w_D \end{cases} \tag{2.5.2}$$

and π_D^* denotes D's equilibrium payoff. It was proven that (2.5.2) never holds if D's resource constraint binds or V's winning benefit is not sufficiently high ($a < b$). Otherwise, one can observe the cases where the settlement is feasible but not desirable. Suppose w_V and w_D affect $\bar{a}$ and $\bar{b}$, respectively. Given the distributions of w_V and w_D, victims, who tend to have less resources than their opponents, do not display strong preferences towards revenge, i.e. ($a < b$) with probability 1.[93] As a result, the ($\pi_D^* > -S$) condition is never satisfied (see Section 2.4). However, perturbing the distribution of V's wealth can reshape individual settlement decisions.[94]

[93] Here, we refer to the case when the coefficients in front of w_V and w_D in the specifications of $\bar{a}$ and $\bar{b}$ differ from zeros.

[94] Alternatively, we could vary the constant term of $\bar{a}$ and get the same effect on the players' behavior.

In the next experiment, we check how players' preferences and behavior change when the allocation of $\bar{w}_V$ varies. Table 2.13 (2.14) reports the results for the N_{all} (N_p) sample. We fix all case-specific characteristics, except w_V. Particularly, a victim who holds the given amount of w_V plays against a set of potential offenders. First, we compute the expected probability to observe a pre-court case resolution for each level of w_V. Then, we identify how often the $(a > b)$ profile and feasible but not desirable settlements appear.

As expected, the increase in V's wealth drives vindictiveness (a) up, and at some point, players' preferences start displaying the $(a > b)$ pattern. However, to achieve this outcome, w_V must grow quite a lot. The probability to settle declines because higher values of a improve V's bargaining position and drive the amount of S up. At the same time, the frequency of feasible but not desirable agreements rises, although they are difficult to support for the given structure of m_V and m_D. Since condition (2.5.2) requires specific combinations of players' preferences and fighting abilities, but only a was perturbed, the latter result is predictable.[95] Overall, one should expect the $(a > b)$ pattern to appear more frequently in disputes where the two parties do not display significant asymmetries in w_V and w_D or the distortion goes in the victim's favor.[96] This happens to be true for other types of crimes and lawsuits (for example, civil litigations where opponents face comparable resource constraints).

With all the observations made, one can discuss "victim-defendant" settlements from the social welfare prospective. Generally, the defendants who can make the offer and want to do so are both richer and have better connections. This also means that their victims display relatively weak fighting abilities and enjoy lower expected vengeance benefits. When we look at a particular "victim-defendant" match, the presence of settlements makes no party worse off in the proposed theoretical setting.[97] However, if the society has preferences that are more than just a sum of V's and D's utilities, the settlements may be abandoned.[98]

So far, we did not specify the objective the policymaker might aim to achieve. In principle, he can have equality concerns and want defendants to face the same relative punishment for a particular type of crime.[99] Allowing for "victim-defendant" settlements, offenders with better fighting abilities encounter milder sanctions (Table 2.6). On top of the income inequality, this generates unfairness in the legal field. Mighty defendants manage to avoid a real punishment through the settlement channel. Also, their victims end up with a lower compensation amount (see Proposition 2.3). Thus, the introduction of settlements can undermine equality before the

[95] See Section 2.4 for more details.

[96] As we explained earlier, in case of criminal traffic offenses the defendants tend to be reacher than their victims. This is especially true when one focuses on "car vs. pedestrian" accidents.

[97] Both V and D obtain their contest equilibrium payoffs at least.

[98] One example comes from incapacitation concerns when the society wants to keep dangerous criminals in prison.

[99] For example, see Fiss (1983).

law, and the policymaker may be willing to declare this institute off.

In 2011–2012, Russian government was considering a possibility to forbid "victim-defendant" settlements for criminal traffic offense with at least one death. The argument against the out-of-court case resolution was exactly the inequality before the law this institute induces. However, the discussion did not result in any changes of the Criminal code.

Further, we illustrate when "victim-defendant" settlements worsen social welfare in the presence of fairness concerns. Suppose the policymaker assigns a value φ to the equality before the law, and his preferences become:

$$SW = \chi_D \sum_{i=1}^{N} u_D^i + \chi_N \sum_{i=1}^{N} u_V^i + \varphi f\left(G_D\right)$$

where

- $\chi_i \geq 0$, $i = \{D, V\}$ denotes how much player i's utility contributes to social welfare;
- G_D reflects the Gini coefficient computed for the distribution of u_D^i;
- N represents a number of observed criminal traffic offenses.

We assume $f\left(G_D\right) = G_D$ and $\chi_D = \chi_N = 1$, i.e. the policymaker equally cares about both sides of the conflict. Now, consider how the presence of "victim-defendant" settlements affects different elements of SW. No private information about the victim's characteristics allows the defendant extract all the surplus when making a settlement offer. Hence, this player obtains more utility if the out-of-court case resolution becomes possible, and the victim is never worse off. As Table 2.6 shows, those defendants who manage to settle with the opponents face much milder punishment than their peers in non-settled cases. Thus, the Gini coefficient rises with the introduction of "victim-defendant" agreements.

Let SW_S (SW_{NS}) denote social welfare when the two conflicting parties can (not) settle among themselves. Also, define $\bar{\varphi}$ as follows:[100]

$$\bar{\varphi}:\ SW_S\left(\bar{\varphi}\right) = SW_{NS}\left(\bar{\varphi}\right)$$

In words, $\overline{\varphi}$ reflects social preferences such that the policymaker is indifferent between banning "victim-defendant" settlements and leaving this practice unchanged. Table 2.8 reports all elements of SW and $\bar{\varphi}$ for two scenarios and various types of criminal traffic offenses. Overall, the introduction of "victim-defendant" settlements allows the policymaker increase $\sum_{i=1}^{N} u_D^i$ by 15.6% (14.5%) for the N_{all} (N_p) sample. At the same time, the Gini coefficient grows by 27.9%

[100]Since SW is linear in φ, the value of $\overline{\varphi}$ must be unique.

Table 2.8: Social Welfare with and without "Victim-Defendant" Settlements

Value	All cases		No deaths		1 death		> 1 deaths	
	S	NS	S	NS	S	NS	S	NS
All cases (N_{all})								
$\sum_{i=1}^{N} u_D^i$	–109.2	–129.4	–59.5	–73.4	–39.9	–45.9	–9.9	–10
$\sum_{i=1}^{N} u_V^i$	.265		.139		.107		.019	
G_D	.472	.369	.483	.366	.441	.354	.361	.347
$\bar{\varphi}$	–197.3		–136.6		–59.5		–1.1	
Car vs. pedestrian (N_p)								
$\sum_{i=1}^{N} u_D^i$	–.649	–.759	–.356	–.435	–.277	–.307	–.017	–.017
$\sum_{i=1}^{N} u_V^i$	3.7E–3		2.1E–3		1.6E–3		9.4E–5	
G_D	.346	.218	.321	.164	.238	.145	.031	.031
$\bar{\varphi}$	–.851		–.618		–.234		0	

Note:
$\sum_{i=1}^{N} u_D^i$, $\sum_{i=1}^{N} u_V^i$ and $\bar{\varphi}$ are measured in E+10 units. The value of $\overline{\varphi}$ corresponds to $SW_S(\bar{\varphi}) = SW_{NS}(\bar{\varphi})$. In the N_p sample, we do not observe settlements for "More than one death" accidents.

(58.7%) in the N_{all} (N_p) case. The strongest inequality corresponds to "No deaths" accidents where the harm made is not so high and the conflicting parties achieve an agreement more often. For any $\varphi < \overline{\varphi} < 0$, the policymaker does not benefit from "victim-defendant" settlements because the cost of inequality becomes significant.[101] Otherwise, the gain in defendants' utility ($\sum_{i=1}^{N} u_D^i$) dominates.

In principle, the optimality of "victim-defendant" settlements also depends on weights the policymaker assigns to $\sum_{i=1}^{N} u_D^i$ and $\sum_{i=1}^{N} u_V^i$ (namely, χ_D and χ_N). Notice that for $\chi_V \geq \chi_D = 0$ and $\varphi < 0$, the society will never allow for out-of-court agreements ($SW_S < SW_{NS}$). Thus, when χ_D is relatively low, "victim-defendant" settlements will make the policymaker worse off even for $|\varphi|$ small enough.

Another argument against "victim-defendant" settlements in the presence of asymmetric bargaining positions relates to deterrence concerns. If advantaged individuals know that in case of a norm violation their victims are likely to have worse fighting abilities, the settlement becomes cheaper. Consequently, they get stronger incentives to break the law than their less advantaged peers. As a result, the settlements make it more problematic to sustain uniform deterrence across different socio-economic groups.

The deterrence concerns may be less important in case of accidental crimes, such as traffic

[101] The negative value of φ can reflect the cost of redistribution associated with growing inequality.

offenses. However, they turn to be crucial when one focuses on intentional felonies. Now, offenders can decide which victim to target. Since individuals with lower income or / and weaker connections are easier to settle with, they are more likely to become victims. Roughly speaking, the presence of pre-court agreements in the criminal law may create a "market" for potential victims. This argument can also convince the policymaker against the given institution.

2.6 Conclusion

Most states use "victim-defendant" settlements to solve civil and criminal conflicts. This paper explores how bargaining positions of the parties involved (namely, their preferences, non-monetary fighting abilities and resource constraints) define the case outcome. Also, we discuss the effect "victim-defendant" settlements may have on social welfare. With this approach, the previous work devoted to out-of-court case resolution connects to the literature that focuses on resource imbalances and the inequality before the law.

We construct a stylized theoretical model where two individuals with conflicting interests, the victim and the defendant, must exert effort in order to achieve / avert the court stage. The defendant has an option to settle with the victim before the fight starts, and the optimal offer decreases in his bargaining position. Reaching the agreement is always efficient when the defendant encounters sufficiently high winning benefits. If the victim displays strong preferences for revenge, but the opponent has better fighting abilities, the latter player is willing to enter the contest stage. Hence, even feasible settlements can fail to happen.

To estimate the model, we employ the data on criminal traffic offenses in Russia and restore bargaining positions of the conflicting parties. Our theoretical framework successfully replicates the observed case outcomes where the key states are "settled", "in court", and "in court & real sentence". On average, defendants have 10 times more resources to expend than victims. At the same time, winning benefits of both parties increase in their wealth. Victims who happen to be close relatives of their offenders have weaker preferences for revenge. Finally, to capture the difference in non-monetary fighting abilities, the estimator needs to be adjusted for non-random selection of law enforcers and government officials into the sample.

Settling with the opponent results in much lower disutility than going to court. Hence, on top of resource imbalances, "victim-defendant" settlements increase the inequality before the law, which may go against the interests of the society. Our counterfactual experiments show that forbidding "victim-defendant" settlements would add more than 2'850 prisoners and cost Russia €2.3 million per year. Also, the frequency of feasible but not desirable agreements rises when we change the wealth distribution for both conflicting parties and victims obtain a pronounced resource advantage.

Although we focused on criminal traffic offenses, the model and the estimation approach proposed in the paper turn to be very general. To push the analysis further, one must specify the objective function of the society and concentrate on the optimal design of the justice system. The criterion may include deterrence and incapacitation concerns, as well as equality considerations. Without this step, it is impossible to give a precise answer when "victim-defendant" settlements must be abandoned, and we leave it for the future.

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Appendix A: Proofs

Proposition 2.1. *The equilibrium of the contest stage exists and is unique.*

Proof. First, consider the unconstrained versions of players' problems.[102] First-order conditions look as follows:

$$\begin{aligned} V: &\quad axp^h \tfrac{e_D}{(e_D+e_V)^2} - m_V = 0 \\ D: &\quad bxp^h \tfrac{e_V}{(e_D+e_V)^2} - m_D = 0 \end{aligned}$$

Notice that second-order derivatives of $\pi_D(\cdot)$ and $\pi_V(\cdot)$ are always negative, and any e_V and e_D that satisfy first-order conditions correspond to an interior maximum. Solving the system of FOCs delivers

$$\begin{gathered} e_V^* = \tfrac{xp^h a^2 m_D b}{(am_D+bm_V)^2},\ e_D^* = \tfrac{xp^h b^2 m_V a}{(am_D+bm_V)^2} \\ \pi_V^* = am_D \tfrac{a^2 m_D x p^h}{(am_D+bm_V)^2} \\ \pi_D^* = -\tfrac{am_D bxp^h}{(am_D+bm_V)^2}\left(am_D + 2bm_V\right) \end{gathered}$$

where asterisks denote equilibrium effort levels and expected payoffs. By construction, this equilibrium is unique and features pure strategies. Now, bring budget constraints back and write down complete first-order conditions:

$$\begin{aligned} V: &\quad axp^h \tfrac{e_D}{(e_D+e_V)^2} - m_V - \lambda_V m_V + \eta_V = 0 \\ D: &\quad bxp^h \tfrac{e_V}{(e_D+e_V)^2} - m_D - \lambda_D m_D + \eta_D = 0 \end{aligned}$$

Here, $\lambda_i \geq 0$ and $\eta_i \geq 0$, $i = \{V, D\}$ are Lagrange multipliers corresponding to $\{w_i - m_i e_i \geq 0\}$ and $\{e_i \geq 0\}$, respectively. The solution of the unconstrained problem, e_V^* and e_D^*, is feasible if and only if

$$\begin{cases} w_V \geq m_V \tfrac{xp^h a^2 m_D b}{(am_D+bm_V)^2} \\ w_D \geq m_D \tfrac{xp^h b^2 m_V a}{(am_D+bm_V)^2} \end{cases} \tag{2.6.1}$$

and this can be supported with $\lambda_i = \eta_i = 0$, $i = \{V, D\}$. Hence, as long as condition (2.6.1) holds, the equilibrium of the contest stage coincides with the one of the unconstraint problem.

Next, we analyze all the cases when at least one budget constraint becomes active.

1. e_D^* is not feasible, and D's budget constraint can bind:

[102] This case is well-studies in the contest literature. With the given specification of the Tullock contest success function, the solution is interior and unique.

$$\begin{cases} w_V \geq m_V \frac{xp^h a^2 m_D b}{(am_D+bm_V)^2} \\ w_D < m_D \frac{xp^h b^2 m_V a}{(am_D+bm_V)^2} \end{cases} \tag{2.6.2}$$

Now, D's optimization program has a corner solution, and his strategy space reduces to $e_D = \left\{0,\, \frac{w_D}{m_D}\right\}$. Suppose in equilibrium D plays $e_D = \frac{w_D}{m_D}$, the highest effort available. Next, assume V's best reply to $\hat{e}_D = \frac{w_D}{m_D}$ solves her first-order condition. Then:

$$\hat{e}_V = \sqrt{\frac{w_D axp^h}{m_D m_V}} - \frac{w_D}{m_D},\ \hat{e}_D = \frac{w_D}{m_D}$$

$$\lambda_D = \frac{bxp^h}{m_D}\frac{\hat{e}_V}{(\hat{e}_V+\hat{e}_D)^2} - 1,\ \eta_D = 0$$

$$\hat{\pi}_V = axp^h - 2\sqrt{\frac{w_D m_V axp^h}{m_D}} + \frac{m_V w_D}{m_D}$$

$$\hat{\pi}_D = -bxp^h + \sqrt{\frac{w_D m_V xp^h}{m_D a}}b - w_D$$

When condition (2.6.2) holds:

- $\lambda_D > 0$ and
- $\hat{e}_V$ is positive and feasible ($m_V \hat{e}_V \leq w_V$), i.e. $\lambda_V = \eta_V = 0$.

Since $\pi_V(\cdot)$ displays strict concavity, $\hat{e}_V$ corresponds to an interior maximum of V's program. If D chooses $e_D = 0$, his equilibrium payoff reaches $\pi_D(0,\, 0) = -bxp^h$, and $\hat{\pi}_D > \pi_D(0,\, 0)$ under condition (2.6.2).[103] Thus, $e_D = \frac{w_D}{m_D}$ strictly dominates $e_D = 0$, and $(\hat{e}_V,\, \hat{e}_D)$ constitutes a unique pure strategy equilibrium of the contest stage when D's budget constraint binds.

2. e_V^* is not feasible, and V's budget constraint can bind:

$$\begin{cases} w_V < m_V \frac{xp^h a^2 m_D b}{(am_D+bm_V)^2} \\ w_D \geq m_D \frac{xp^h b^2 m_V a}{(am_D+bm_V)^2} \end{cases} \tag{2.6.3}$$

The analysis employs all the arguments developed in point 1. V has two options to choose: $e_V = \frac{w_V}{m_V}$ and $e_V = 0$. Assume in equilibrium V plays $e_V = \frac{w_V}{m_V}$, and D's best reply comes from his first-order condition:

[103] Also, the case of $\lambda_D = 0$, $\eta_D > 0$ does not deliver well-defined Lagrange multipliers and results in a contradiction.

$$\hat{e}_V = \tfrac{w_V}{m_V},\ \hat{e}_D = \sqrt{\tfrac{w_V bxp^h}{m_D m_V}} - \tfrac{w_V}{m_V}$$
$$\lambda_V = \tfrac{axp^h}{m_V}\tfrac{\hat{e}_D}{(\hat{e}_V+\hat{e}_D)^2} - 1,\ \eta_V = 0$$
$$\hat{\pi}_V = \sqrt{\tfrac{w_V m_D xp^h}{m_V b}}a - w_V$$
$$\hat{\pi}_D = -2\sqrt{\tfrac{w_V m_D bxp^h}{m_V}} + \tfrac{m_D w_V}{m_V}$$

where $\lambda_V > 0$, $\hat{e}_D$ maximizes $\pi_D(\cdot)$ and satisfies D's budget constraint under condition (2.6.3). It is easy to show that $\pi_V(0,\, e_D) < \hat{\pi}_V$ and $e_V = \frac{w_V}{m_V}$ strictly dominates $e_V = 0$. Hence, $(\hat{e}_V,\, \hat{e}_D)$ is a unique pure strategy equilibrium of the contest game.

3. Neither e_V^* nor e_D^* are feasible, and budget constraints of both players can bind:

$$\begin{cases} w_V < m_V\frac{xp^h a^2 m_D b}{(am_D+bm_V)^2} \\ w_D < m_D\frac{xp^h b^2 m_V a}{(am_D+bm_V)^2} \end{cases} \tag{2.6.4}$$

In this case, the contestants can no longer afford the solution of the unconstrained program. Players can exert either zero effort or expend all resources available, i.e. $e_i = \left\{0,\, \frac{w_i}{m_i}\right\}$, $i = \{V,\, D\}$. The first-order conditions of contestants' programs do not behave well if $e_V = 0$ or / and $e_D = 0$.[104] For this reason, we work with the payoff matrix directly:

D		V	
		$e_V = 0$	$e_V = \frac{w_V}{m_V}$
	$e_D = 0$	$(-bxp^h,\, axp^h)$	$(-bxp^h,\, axp^h - w_V)$
	$e_D = \frac{w_D}{m_D}$	$(-w_D,\, 0)$	$\left(-bxp^h P_C\left(\frac{w_V}{m_V},\, \frac{w_D}{m_D}\right) - w_D,\, axp^h P_C\left(\frac{w_V}{m_V},\, \frac{w_D}{m_D}\right) - w_V\right)$

If D chooses $\{e_D = 0\}$, V responds with $\{e_V = 0\}$ as well. When V plays $\{e_V = 0\}$, D is better off exerting $\left\{e_D = \frac{w_D}{m_D}\right\}$. Best replies to $\left\{e_D = \frac{w_D}{m_D}\right\}$ and $\left\{e_V = \frac{w_V}{m_V}\right\}$ depend on w_D and w_V:

- $w_V < axp^h - \frac{m_V}{m_D}w_D \Rightarrow V$ prefers $\left\{e_V = \frac{w_V}{m_V}\right\}$ to $\{e_V = 0\}$ when D plays $\left\{e_D = \frac{w_D}{m_D}\right\}$
- $w_V < \frac{m_V}{m_D}\left(bxp^h - w_D\right) \Rightarrow D$ prefers $\left\{e_D = \frac{w_D}{m_D}\right\}$ to $\{e_D = 0\}$ when V plays $\left\{e_V = \frac{w_V}{m_V}\right\}$

Now, we show that (2.6.4) implies $w_V < \min\left\{axp^h,\, \frac{m_V}{m_D}bxp^h\right\} - \frac{m_V}{m_D}w_D$. Suppose $w_V \geq axp^h - \frac{m_V}{m_D}w_D$ is compatible with (2.6.4). Then, the following set must be non-empty:

$$\left[axp^h - \frac{m_V}{m_D}w_D,\, m_V\frac{xp^h a^2 m_D b}{(am_D+bm_V)^2}\right) \neq \emptyset \Leftrightarrow w_D > axp^h\frac{m_D}{m_V} - \frac{xp^h a^2 m_D^2 b}{(am_D+bm_V)^2} \tag{2.6.5}$$

[104]When $e_V = 0$ or / and $e_D = 0$, there do not exist well-defined Lagrange multipliers that can support an interior solution.

Also, condition (2.6.5) defines a non-empty intersection with (2.6.4) if and only if

$$axp^h \frac{m_D}{m_V} - \frac{xp^h a^2 m_D^2 b}{(am_D + bm_V)^2} < m_D \frac{xp^h b^2 m_V a}{(am_D + bm_V)^2} \Leftrightarrow am_D (am_D + bm_V) < 0 \tag{2.6.6}$$

where the latter results in a contradiction. Hence, $w_V \geq axp^h - \frac{m_V}{m_D} w_D$ never combines with (2.6.4), and (2.6.4) must imply $w_V < axp^h - \frac{m_V}{m_D} w_D$. Similarly, one can prove that $w_V \geq \frac{m_V}{m_D} \left(bxp^h - w_D \right)$ and (2.6.4) are disjoint. Thus, under condition (2.6.4), D must have a dominant strategy $\left\{ e_D = \frac{w_D}{m_D} \right\}$. Then, the unique equilibrium is

$$\begin{gathered}
\hat{e}_V = \tfrac{w_V}{m_V},\ \hat{e}_D = \tfrac{w_D}{m_D} \\
\lambda_V = \tfrac{axp^h}{m_V} \tfrac{\hat{e}_D}{(\hat{e}_V + \hat{e}_D)^2} - 1,\ \lambda_D = \tfrac{bxp^h}{m_D} \tfrac{\hat{e}_V}{(\hat{e}_V + \hat{e}_D)^2} - 1 \\
\eta_V = \eta_D = 0 \\
\hat{\pi}_V = axp^h P_C \left(\tfrac{w_V}{m_V}, \tfrac{w_D}{m_D} \right) - w_V \\
\hat{\pi}_D = -bxp^h P_C \left(\tfrac{w_V}{m_V}, \tfrac{w_D}{m_D} \right) - w_D
\end{gathered}$$

Under all conditions imposed on contestants' resources, λ_V and λ_D are strictly positive and support an interior solution.

$\square$

Proposition 2.2. *Contestants' equilibrium effort, e_i^*, $i = \{V, D\}$ always increases in his / her valuation of punishment and w_i, decreases in m_i:*

$$\frac{\partial e_V^*}{\partial a} \geq 0,\ \frac{\partial e_D^*}{\partial b} \geq 0,\ \frac{\partial e_i^*}{\partial w_i} \geq 0,\ \frac{\partial e_i^*}{\partial m_i} \leq 0,\ i = \{V, D\}$$

For $\frac{a}{m_V} \geq \frac{b}{m_D}$

1. *e_V^* increases in b and e_D^* decreases in a*
2. *e_V^* decreases in m_D and increases in w_D*
3. *e_D^* increases in m_V and decreases in w_V if and only if $w_V \geq \frac{bxp^h m_V}{4m_D} > 0$. Otherwise, e_D^* strictly decreases in m_V and strictly increases in w_V*

For $\frac{a}{m_V} < \frac{b}{m_D}$

1. *e_V^* strictly decreases in b and e_D^* strictly increases in a*
2. *e_V^* increases in m_D and decreases in w_D if and only if $w_D \geq \frac{axp^h m_D}{4m_V} > 0$. Otherwise, e_V^* strictly decreases in m_D and strictly increases in w_D*
3. *e_D^* strictly decreases in m_V and strictly increases in w_V*

Proof. To show how contestants' effort choice depends on their fighting abilities and preferences, we inspect all possible equilibrium outcomes. First, check how e_D^* and e_V^* change with b and a, respectively:

$$\begin{aligned}
UC:&\quad \frac{\partial e_V^*}{\partial a} = \frac{2axp^h b^2 m_D m_V}{(am_D + bm_V)^3} > 0,\ \frac{\partial e_D^*}{\partial b} = \frac{2bxp^h a^2 m_D m_V}{(am_D + bm_V)^3} > 0 \\
BC_D:&\quad \frac{\partial e_V^*}{\partial a} = \frac{1}{2}\sqrt{\frac{w_D xp^h}{am_D m_V}} > 0,\ \frac{\partial e_D^*}{\partial b} = 0 \\
BC_V:&\quad \frac{\partial e_V^*}{\partial a} = 0,\ \frac{\partial e_D^*}{\partial b} = \frac{1}{2}\sqrt{\frac{w_V xp^h}{bm_D m_V}} > 0 \\
BC_{VD}:&\quad \frac{\partial e_V^*}{\partial a} = \frac{\partial e_D^*}{\partial b} = 0
\end{aligned}$$

where UC denotes the unconstrained problem; BC_D (BC_V) defines the situation when D's (V's) budget constraint binds; in case of BC_{VD} the solution of UC is no longer feasible for both contestants. Hence, players' equilibrium effort never decreases in their valuations punishment, i.e. $\frac{\partial e_V^*}{\partial a} \geq 0$, $\frac{\partial e_D^*}{\partial b} \geq 0$.

Second, we investigate the effect of m_i on e_i^* and e_j^*, $i, j = \{V, D\}$, $i \neq j$:

$$
\begin{aligned}
UC: \quad & \frac{\partial e_V^*}{\partial m_V} = -\frac{2a^2xp^h b^2 m_D}{(am_D+bm_V)^3} < 0,\ \frac{\partial e_D^*}{\partial m_D} = -\frac{2b^2xp^h a^2 m_V}{(am_D+bm_V)^3} < 0 \\
& \frac{\partial e_V^*}{\partial m_D} = -\frac{a^2xp^h b(am_D-bm_V)}{(am_D+bm_V)^3},\ \frac{\partial e_D^*}{\partial m_V} = \frac{b^2xp^h a(am_D-bm_V)}{(am_D+bm_V)^3} \\
BC_D: \quad & \frac{\partial e_V^*}{\partial m_V} = -\frac{1}{2m_V}\sqrt{\frac{w_D axp^h}{m_D m_V}} < 0,\ \frac{\partial e_D^*}{\partial m_D} = -\frac{w_D}{m_D^2} < 0 \\
& \frac{\partial e_V^*}{\partial m_D} = -\frac{1}{2m_D}\sqrt{\frac{w_D axp^h}{m_D m_V}} + \frac{w_D}{m_D^2},\ \frac{\partial e_D^*}{\partial m_D} = 0 \\
BC_V: \quad & \frac{\partial e_V^*}{\partial m_V} = < 0,\ \frac{\partial e_D^*}{\partial m_D} = -\frac{1}{2m_D}\sqrt{\frac{w_V bxp^h}{m_D m_V}} < 0 \\
& \frac{\partial e_V^*}{\partial m_D} = 0,\ \frac{\partial e_D^*}{\partial m_V} = -\frac{1}{2m_V}\sqrt{\frac{w_V bxp^h}{m_D m_V}} + \frac{w_V}{m_D^2} \\
BC_{VD}: \quad & \frac{\partial e_V^*}{\partial m_V} = -\frac{w_V}{m_V^2},\ \frac{\partial e_D^*}{\partial m_D} = -\frac{w_D}{m_D^2} \\
& \frac{\partial e_V^*}{\partial m_D} = \frac{\partial e_D^*}{\partial m_V} = 0
\end{aligned}
$$

Contestants' equilibrium effort e_i^* increases in m_i for any preference profile. The effect of m_j on e_i^*, $i \neq j$ is ambiguous. Take the UC case. If $\frac{a}{m_V} \geq \frac{b}{m_D}$, it must be $\frac{\partial e_V^*}{\partial m_D} \leq 0$, $\frac{\partial e_D^*}{\partial m_V} \geq 0$, and the opposite holds for $\frac{a}{m_V} < \frac{b}{m_D}$. Next, consider the BC_D scenario, which also requires condition (2.6.2) from the proof of Proposition 1. The derivative $\frac{\partial e_V^*}{\partial m_D}$ is non-negative if and only if

$$\frac{\partial e_V^*}{\partial m_D} \geq 0 \Leftrightarrow w_D \geq \frac{axp^h m_D}{4m_V} \tag{2.6.7}$$

Otherwise, $\frac{\partial e_V^*}{\partial m_D} < 0$ holds. The inequality (2.6.7) defines a non-empty intersection with (2.6.2) if and only if $\frac{a}{m_V} < \frac{b}{m_D}$. Otherwise, e_V^* decreases in m_D. In the BC_V case, e_D^* (weakly) increases in m_V if and only if

$$\frac{\partial e_D^*}{\partial m_V} \geq 0 \Leftrightarrow w_V \geq \frac{bxp^h m_V}{4m_D} \tag{2.6.8}$$

Condition (2.6.3) supports the BC_V scenario. It has a non-empty intersection with (2.6.8) if and only if $\frac{a}{m_V} \geq \frac{b}{m_D}$. Combining the results obtained for different equilibrium outcomes and "preferences–fighting abilities" profiles, we get the effects of m_i on e_i^* and e_j^*, $i \neq j$ as stated in the proposition.

Finally, compute the derivatives of e_i^* with respect to w_i and w_j, $i \neq j$:

$$
\begin{aligned}
UC: \quad & \frac{\partial e_i^*}{\partial w_i} = \frac{\partial e_i^*}{\partial w_j} = 0,\ i,\ j = \{V,\ D\},\ i \neq j \\
BC_D: \quad & \frac{\partial e_V^*}{\partial w_V} = 0,\ \frac{\partial e_D^*}{\partial w_D} = \frac{1}{m_D} > 0 \\
& \frac{\partial e_V^*}{\partial w_D} = \frac{1}{2}\sqrt{\frac{axp^h}{m_D m_V w_D}} - \frac{1}{m_D},\ \frac{\partial e_D^*}{\partial w_V} = 0
\end{aligned}
$$

$$\begin{array}{ll} BC_V: & \frac{\partial e_V^*}{\partial w_V} = \frac{1}{m_V} > 0,\ \frac{\partial e_D^*}{\partial w_D} = 0 \\ & \frac{\partial e_V^*}{\partial w_D} = 0,\ \frac{\partial e_D^*}{\partial w_V} = \frac{1}{2}\sqrt{\frac{bxp^h}{m_D m_V w_V}} - \frac{1}{m_V} \\ BC_{VD}: & \frac{\partial e_V^*}{\partial w_V} = \frac{1}{w_V},\ \frac{\partial e_D^*}{\partial w_D} = \frac{1}{w_D} \\ & \frac{\partial e_V^*}{\partial w_D} = \frac{\partial e_D^*}{\partial w_V} = 0 \end{array}$$

The UC equilibrium effort levels display no response to w_V and w_D because the constraints do not bite. Look at the BC_D case. V's equilibrium effort strictly increases in w_D if and only if

$$\frac{\partial e_V^*}{\partial w_D} > 0 \Leftrightarrow w_D < \frac{axp^h m_D}{4m_V} \tag{2.6.9}$$

When $\frac{a}{m_V} \geq \frac{b}{m_D}$, condition (2.6.2) implies (2.6.9), and e_V^* always increases in w_D under the BC_D scenario. Otherwise, $\frac{\partial e_V^*}{\partial w_D} \leq 0$ holds for any $w_D \geq \frac{axp^h m_D}{4m_V}$. Next, turn to the BC_V case. The effort D exerts in equilibrium strictly increases in w_V if and only if

$$\frac{\partial e_D^*}{\partial w_V} > 0 \Leftrightarrow w_V < \frac{bxp^h m_V}{4m_D} \tag{2.6.10}$$

and this is always satisfied for $\frac{a}{m_V} < \frac{b}{m_D}$ in the BC_V scenario. If $\frac{a}{m_V} \geq \frac{b}{m_D}$, we observe $\frac{\partial e_D^*}{\partial w_V} \leq 0$ for any $w_V \geq \frac{bxp^h m_V}{4m_D}$. Putting things together, the effect of w_i and w_j on e_i^*, $i \neq j$ follows.

□

Proposition 2.3. *The optimal settlement offer S always decreases (increases) in D's (V's) willingness to win b (a) and his fighting abilities. S always increases in V's non-monetary fighting ability. S increases in w_V if and only if w_V is sufficiently small ($w_V \in [0,\, \tilde{w}_V]$, $\tilde{w}_V > 0$).*

Proof. To prove the claim, recall Lemma 1, which states that the optimal settlement offer S equals to V's equilibrium payoff π_V^*. Hence, π_V^*'s comparative statics coincide with those of S. Consider all possible equilibrium outcomes. First, take the case when contestants' budget constraints do not bind and V's equilibrium payoff reaches

$$\pi_V^* = am_D \frac{a^2 m_D x p^h}{(am_D + bm_V)^2}$$

$\pi_V^* \equiv S$ responds to changes in players' willingness to win and fighting abilities as follows:

$$\frac{\partial \pi_V^*}{\partial a} = \frac{a^2 m_D x p^h}{(am_D+bm_V)^3}\,(am_D + 3bm_V) > 0,\ \frac{\partial \pi_V^*}{\partial b} = -\frac{2a^3 m_D^2 m_V x p^h}{(am_D+bm_V)^3} < 0$$
$$\frac{\partial \pi_V^*}{\partial m_V} = -\frac{2a^3 m_D^2 b x p^h}{(am_D+bm_V)^3} < 0,\ \frac{\partial \pi_V^*}{\partial m_D} = \frac{2a^3 m_D m_V b x p^h}{(am_D+bm_V)^3} > 0$$
$$\frac{\partial \pi_V^*}{\partial w_V} = \frac{\partial \pi_V^*}{\partial w_D} = 0$$

where non-binding budget constraints imply no effect of w_i, $i = \{V,\, D\}$ on π_V^*.

Next, we investigate the equilibrium where D plays $e_D = \frac{w_D}{m_D}$ and V's best reply comes from her first-order condition:

$$\pi_V^* = axp^h - 2\sqrt{\frac{w_D m_V a x p^h}{m_D}} + \frac{m_V w_D}{m_D}$$
$$\frac{\partial \pi_V^*}{\partial a} = xp^h - \sqrt{\frac{w_D m_V x p^h}{am_D}},\ \frac{\partial \pi_V^*}{\partial b} = 0$$
$$\frac{\partial \pi_V^*}{\partial m_V} = \frac{w_D}{m_D} - \sqrt{\frac{w_D a x p^h}{m_D m_V}},\ \frac{\partial \pi_V^*}{\partial m_D} = -\frac{m_V w_D}{m_D^2} + \frac{1}{m_D}\sqrt{\frac{w_D m_V a x p^h}{m_D}}$$
$$\frac{\partial \pi_V^*}{\partial w_V} = 0,\ \frac{\partial \pi_V^*}{\partial w_D} = \frac{m_V}{m_D} - \sqrt{\frac{m_V a x p^h}{m_D w_D}}$$

and the signs of these derivatives are defined by

$$w_D < \frac{axp^h m_D}{m_V} \tag{2.6.11}$$

The given equilibrium outcomes requires condition (2.6.2) from the proof of Proposition 2.1, and (2.6.2) implies (2.6.11). Hence, the effects of a, m_V, m_D, and w_D on π_V^* become unambiguous:

$$\frac{\partial \pi_V^*}{\partial a} > 0,\ \frac{\partial \pi_V^*}{\partial m_V} < 0,\ \frac{\partial \pi_V^*}{\partial m_D} > 0,\ \frac{\partial \pi_V^*}{\partial w_D} < 0$$

When V's budget constraint binds, but D still has enough resources, the victim obtains

$$\pi_V^* = \sqrt{\frac{w_V m_D x p^h}{m_V b}a} - w_V$$

The effects of interest are

$$\frac{\partial\pi_V^*}{\partial a}=\sqrt{\frac{w_V m_D x p^h}{m_V b}}>0,\ \frac{\partial\pi_V^*}{\partial b}=-\frac{a}{2b}\sqrt{\frac{w_V m_D x p^h}{m_V b}}<0$$
$$\frac{\partial\pi_V^*}{\partial m_V}=-\frac{a}{2m_V}\sqrt{\frac{w_V m_D x p^h}{m_V b}}<0,\ \frac{\partial\pi_V^*}{\partial m_D}=-\frac{a}{2}\sqrt{\frac{w_V x p^h}{m_D m_V b}}>0$$
$$\frac{\partial\pi_V^*}{\partial w_V}=\frac{a}{2}\sqrt{\frac{m_D x p^h}{w_V m_V b}}-1>0\Leftrightarrow w_V<\frac{a^2xp^hm_D}{4bm_V},\ \frac{\partial\pi_V^*}{\partial w_D}=0$$

To support this equilibrium configuration, condition (2.6.3) from the proof of Proposition 2.1 is necessary. If $\frac{a}{m_V}\geq\frac{b}{m_D}$, (2.6.3) implies $w_V<\frac{a^2xp^hm_D}{4bm_V}$, and $\frac{\partial\pi_V^*}{\partial w_V}>0$ always holds. Otherwise, $\frac{\partial\pi_V^*}{\partial w_V}>0$ for $w_V<\frac{a^2xp^hm_D}{4bm_V}$ and $\frac{\partial\pi_V^*}{\partial w_V}\leq 0$ for $w_V\in\left[\frac{a^2xp^hm_D}{4bm_V},\, m_V\frac{xp^ha^2m_Db}{(am_D+bm_V)^2}\right)$. Take

$$\tilde{w}_V=\min\left\{\frac{a^2xp^hm_D}{4bm_V},\, m_V\frac{xp^ha^2m_Db}{(am_D+bm_V)^2}\right\}$$

and the claim of the proposition follows.

Finally, we study the case when both contestants face tight budget constraints (condition (2.6.4) from the proof of Proposition 2.1):

$$\pi_V^*=axp^hP_C\left(\frac{w_V}{m_V},\frac{w_D}{m_D}\right)-w_V$$
$$\frac{\partial\pi_V^*}{\partial a}=xp^hP_C\left(\frac{w_V}{m_V},\frac{w_D}{m_D}\right)>0,\ \frac{\partial\pi_V^*}{\partial b}=0$$
$$\frac{\partial\pi_V^*}{\partial m_V}=-axp^h\frac{w_Vw_Dm_D}{(w_Vm_D+w_Dm_V)^2}<0,\ \frac{\partial\pi_V^*}{\partial m_D}=axp^h\frac{w_Dw_Vm_V}{(w_Vm_D+w_Dm_V)^2}>0$$
$$\frac{\partial\pi_V^*}{\partial w_V}=axp^h\frac{w_Dm_Vm_D}{(w_Vm_D+w_Dm_V)^2}-1>0\Leftrightarrow\sum_{i=1}^2\frac{w_i}{m_i}<\sqrt{\frac{axp^hw_D}{m_Vm_D}}$$
$$\frac{\partial\pi_V^*}{\partial w_D}=-axp^h\frac{w_Vm_Dm_V}{(w_Vm_D+w_Dm_V)^2}<0$$

Define $\tilde{w}_V=\min\left\{\sqrt{\frac{axp^hw_D}{m_Vm_D}}m_V-\frac{m_V}{m_D}w_D,\, m_V\frac{xp^ha^2m_Db}{(am_D+bm_V)^2}\right\}$ and get the statement of the proposition.

$\square$

Proposition 2.4. *There exist non-empty sets of "preference–abilities" profiles $Y_{\bar{S}} \subset Y_{a>b}$ and $Y_S \subset Y_{a>b}$ such that*

- *For any $y \in Y_{\bar{S}}$ the defendant has enough resources to settle but is not willing to do so:*

$$\begin{cases} -xp^h P_C\left(e_V^*,\, e_D^*\right)(a-b) + m_V e_V^* + m_D e_D^* < 0 \\ axp^h P_C\left(e_V^*,\, e_D\right) - m_V e_V^* \leq w_D \end{cases} \neq \emptyset$$

- *For any $y \in Y_S$ the defendant has enough resources to settle and is willing to do so:*

$$\begin{cases} -xp^h P_C\left(e_V^*,\, e_D^*\right)(a-b) + m_V e_V^* + m_D e_D^* \geq 0 \\ axp^h P_C\left(e_V^*,\, e_D\right) - m_V e_V^* \leq w_D \end{cases} \neq \emptyset$$

Proof. To prove the proposition, we analyze all equilibrium outcomes separately. Define $\tilde{w}_D = m_D \frac{xp^h b^2 a m_V}{(am_D+bm_V)^2}$ and $\tilde{w}_V = m\frac{xp^h a^2 eb}{(ae+bm)^2}$. First, take the case when contestants' budget constraints do not bind. Condition (2.6.1) from the proof of Proposition 2.1 supports this scenario. In equilibrium, players obtain

$$\pi_V^* = am_D \tfrac{a^2 m_D xp^h}{(am_D+bm_V)^2}$$
$$\pi_D^* = -\tfrac{am_D bxp^h}{(am_D+bm_V)^2}\left(am_D + 2bm_V\right)$$

The optimal settlement offer is $S = \pi_V^*$ (Lemma 1). The game does not proceed to the contest stage if and only if

$$\begin{cases} \pi_D^* \leq -S \\ S \leq w_D \end{cases} \Leftrightarrow \begin{cases} m_V \geq \frac{am_D(a-b)}{2b^2} = \hat{m}_V \\ w_D \geq \frac{a^3 m_D^2 xp^h}{(am_D+bm_V)^2} \end{cases}$$

The latter inequality always defines a non-empty intersection with condition (2.6.1). Taking $m_V \in [0,\, \hat{m}_V)$ and $w_D \geq \max\left\{\tilde{w}_D,\, \frac{a^3 m_D^2 xp^h}{(am_D+bm_V)^2}\right\}$, one gets the case when the settlement is feasible, but D strictly prefers to fight:

$$Y_{\bar{S}}^1 = \left\{ y \in Y_{a>b} : m_V \in [0,\, \hat{m}_V)\,,\, w_V \geq \tilde{w}_V,\, w_D \geq \max\left\{\tilde{w}_D,\, \frac{a^3 m_D^2 xp^h}{(am_D+bm_V)^2}\right\}\right\}$$

where $Y_{\bar{S}} \subset Y_{a>b}$. A set of "preference–abilities" profiles such that the settlement indeed happens looks as follows:

$$Y_S^1 = \left\{ y \in Y_{a>b} : m_V \geq \hat{m}_V,\, w_V \geq \tilde{w}_V,\, w_D \geq \max\left\{\tilde{w}_D,\, \frac{a^3 m_D^2 xp^h}{(am_D+bm_V)^2}\right\}\right\}$$

Next, consider the equilibrium where D's budget constraint binds (condition (2.6.2) from the proof of Proposition 2.1 is needed):

$$\pi_V^* = axp^h - 2\sqrt{\tfrac{w_D m_V axp^h}{m_D}} + \tfrac{m_V w_D}{m_D}$$
$$\pi_D^* = -bxp^h + \sqrt{\tfrac{w_D m_V xp^h}{m_D a}} b - w_D$$

D makes a settlement offer if and only if

$$\begin{cases} \pi_D^* \leq -S \\ S \leq w_D \end{cases} \Leftrightarrow \begin{cases} \pi_V^* \leq bxp^h - \sqrt{\tfrac{w_D m_V xp^h}{m_D a}} b + w_D \\ \pi_V^* \leq w_D \end{cases}$$

When condition (2.6.2) holds, it must be $\left\{xp^h - \sqrt{\tfrac{w_D m_V xp^h}{m_D a}} > 0\right\}$. Then, $\{S \leq w_D\}$ implies $\{\pi_D^* \leq -S\}$, i.e. a feasible settlement is always desirable by D. The $\{S \leq w_D\}$ condition holds if and only if

$$w_D \left(\frac{m_D - m_V}{m_D}\right) + 2\sqrt{w_D}\sqrt{\frac{m_V axp^h}{m_D}} - axp^h \geq 0$$

Solving the underlying equation for $\sqrt{w_D}$ delivers two real roots, r_1 and r_2:

$$r_{1,2} = \frac{m_D \left(\pm\sqrt{axp^h} - \sqrt{\tfrac{m_V axp^h}{m_D}}\right)}{m_D - m_V}$$

Depending on m_D and m_V, different cases emerge:

- $m_D > m_V \Rightarrow r_1 > 0$, $r_2 < 0$, and the settlement offer requires $w_D \geq r_1^2$, and this defines a non-empty intersection with condition (2.6.2) if and only if

$$\begin{cases} w_D \geq r_1^2 \\ w_D < m_D \tfrac{xp^h b^2 a m_V}{(am_D + bm_V)^2} \end{cases} \neq \emptyset \Leftrightarrow r_1^2 < m_D \frac{xp^h b^2 a m_V}{(am_D + bm_V)^2} \Leftrightarrow m_V > \frac{a^2 m_D}{b^2}$$

  With $a > b$, the last inequality contradicts $m_D > m_V$, and no settlement offer is made.

- $m_D < m_V \Rightarrow r_1 < 0$, $r_2 > 0$, and the offer appears under $w_D \leq r_2^2$. Hence, V and D settle if and only if $w_D < \min\{r_2^2, \tilde{w}_D\}$:

$$Y_S^2 = \left\{y \in Y_{a>b} : \; m_V > m_D, \, w_V \geq \tilde{w}_V, \, w_D < \min\left\{r_2^2, \, \tilde{w}_D\right\}\right\}$$

Further, we analyze the case when only V's budget constraint binds (condition (2.6.3) from the proof of Proposition 2.1):

$$\pi_V^* = \sqrt{\tfrac{w_V m_D xp^h}{m_V b}} a - w_V$$
$$\pi_D^* = -2\sqrt{\tfrac{w_V m_D bxp^h}{m_V}} + \tfrac{m_D w_V}{m_V}$$

It is optimal to settle if and only if

$$\begin{cases} \pi_D^* \leq -S \\ S \leq w_D \end{cases} \Leftrightarrow \begin{cases} \sqrt{w_V}\frac{(m_D - m_V)}{m_V} \leq \sqrt{\frac{m_D x p^h}{m_V b}}\,(2b - a) \\ \sqrt{\frac{w_V m_D x p^h}{m_V b}}a - w_V \leq w_D \end{cases}$$

If D has enough wealth ($w_D \geq \hat{w}_D = \max\left\{\tilde{w}_D,\, \sqrt{\frac{w_V m_V x p^h}{m_D b}}a - w_V\right\}$), the second inequality always holds, i.e. the settlement is feasible. However, the willingness to settle ($\pi_D^* \leq -S$) strongly depends on players' preferences and fighting abilities:

- $m_D > m_V$ (V has an advantage in non-monetary fighting abilities) $\Rightarrow$ two cases emerge:
  - $a \geq 2b$ (V is vindictive enough) $\Rightarrow$ D never wants to settle:

$$\sqrt{w_V} \leq \sqrt{\frac{m_D x p^h}{m_V b}}\frac{(2b - a)\, m_V}{(m_D - m_V)} < 0,\, a\, contradiction$$

$$Y_{\bar{S}}^2 = \{y \in Y_{a>b} : \, m_D > m_V,\, a \geq 2b,\, w_V < \tilde{w}_V,\, w_D \geq \hat{w}_D\}$$

  - $a \in (b,\, 2b)$ $\Rightarrow$ D is willing to settle if and only if

$$w_V < \hat{w}_V = \min\left\{m_D\frac{m_V x p^h\,(2b - a)^2}{b\,(m_D - m_V)^2},\, \tilde{w}_V\right\}$$

    When $m_V < \hat{m}_V$, it must be

$$\min\left\{m_V\frac{m_D x p^h\,(2b - a)^2}{b\,(m_D - m_V)^2},\, \tilde{w}_V\right\} = m_V\frac{m_D x p^h\,(2b - a)^2}{b\,(m_D - m_V)^2} \equiv \bar{w}_V$$

    Then, one can specify non-empty subsets of $Y_{\bar{S}}$ and Y_S:

$$Y_{\bar{S}}^3 = \{y \in Y_{a>b} : \, m_D > m_V,\, m_V < \hat{m}_V,\, a \in (b,\, 2b)\,,\, w_V \in [\hat{w}_V,\, \tilde{w}_V)\,,\, w_D \geq \hat{w}_D\}$$

$$Y_S^3 = \{y \in Y_{a>b} : \, m_D > m_V,\, m_V < \hat{m}_V,\, a \in (b,\, 2b)\,,\, w_V < \hat{w}_V,\, w_D \geq \hat{w}_D\}$$

$$Y_S^4 = \{y \in Y_{a>b} : \, m_D > m_V,\, m_V \geq \hat{m}_V,\, a \in (b,\, 2b)\,,\, w_V < \hat{w}_V,\, w_D \geq \hat{w}_D\}$$

- $m_D < m_V$ (D has an advantage in non-monetary fighting abilities) :
  - $a \geq 2b$ (V is vindictive enough) $\Rightarrow$ D makes an offer if and only if

$$w_V \in (\bar{w}_V,\, \tilde{w}_V)$$

    and this set is non-empty if and only if $m_V > \hat{m}_V$. With this result, non-empty subsets of $Y_{\bar{S}}$ and Y_S are

$$Y_{\bar{S}}^4 = \{y \in Y_{a>b} : \, m_D < m_V, \, m_V \le \hat{m}_V, \, a \ge 2b, \, w_V < \tilde{w}_V, \, w_D \ge \hat{w}_D\}$$

$$Y_{\bar{S}}^5 = \{y \in Y_{a>b} : \, m_D < m_V, \, m_V > \hat{m}_V, \, a \ge 2b, \, w_V \le \bar{w}_V, \, w_D \ge \hat{w}_D\}$$

$$Y_S^5 = \{y \in Y_{a>b} : \, m_D < m_V, \, m_V > \hat{m}_V, \, a \ge 2b, \, w_V \in (\bar{w}_V, \, \tilde{w}_V), \, w_D \ge \hat{w}_D\}$$

– If V does not get sufficient benefits from D being punished ($a \in (b, \, 2b)$), the defendant always prefers to settle:

$$Y_S^6 = \{y \in Y_{a>b} : \, m_D < m_V, \, a \in (b, \, 2b), \, w_V < \tilde{w}_V, \, w_D \ge \hat{w}_D\}$$

Finally, check the equilibrium where both contestants face binding budget constraints (condition (2.6.4) from the proof of Proposition 2.1):

$$\pi_V^* = axp^h P_C\left(\tfrac{w_V}{m_V}, \, \tfrac{w_D}{m_D}\right) - w_V$$
$$\pi_D^* = -bxp^h P_C\left(\tfrac{w_V}{m_V}, \, \tfrac{w_D}{m_D}\right) - w_D$$

The settlement requires

$$\begin{cases} \pi_D^* \le -S \\ S \le w_D \end{cases} \Leftrightarrow \begin{cases} axp^h P_C\left(\tfrac{w_V}{m_V}, \, \tfrac{w_D}{m_D}\right) - w_V \le bxp^h P_C\left(\tfrac{w_V}{m}, \, \tfrac{w_D}{e}\right) + w_D \\ axp^h P_C\left(\tfrac{w_V}{m_V}, \, \tfrac{w_D}{m_D}\right) - w_V \le w_D \end{cases}$$

where the latter inequality implies the former one. Thus, if the settlement is feasible, D does not want to move to the contest stage. One can reduce the second condition to

$$w_V^2 m_D + \left(w_D \left(m_D + m_V\right) - axp^h m_D\right) w_V + w_D^2 m_V \ge 0 \tag{2.6.12}$$

If (2.6.4) holds, it must be $\left\{w_D \left(m_D + m_V\right) - axp^h < 0\right\}$, and (2.6.12) may be violated. When we solve (2.6.12) with respect to w_V, two possibilities appear:

- The discriminant of the underlying square equation is non-negative $\Rightarrow$ there are two real roots, $\tilde{r}_1$ and $\tilde{r}_2$, $0 < \tilde{r}_1 \le \tilde{r}_2$. Then, (2.6.12) is satisfied for any $w_V \in [0, \, \tilde{r}_1] \cup [\tilde{r}_2, \, \infty)$, and we can define a non-empty subset of Y_S:

$$Y_S^6 = \{y \in Y_{a>b} : \, w_V < \min\{\tilde{r}_1, \, \tilde{w}_V\}, \, w_D < \tilde{w}_D\}$$

- The discriminant of the underlying square equation is negative $\Rightarrow$ (2.6.12) always holds:

$$Y_S^7 = \{y \in Y_{a>b} : \, w_V < \tilde{w}_V, \, w_D < \tilde{w}_D\}$$

Finally, define $Y_{\bar{S}}$ and Y_S as follows:

$$Y_{\bar{S}} = \cup_{i=1}^5 Y_{\bar{S}}^i, \; Y_S = \cup_{i=1}^7 Y_S^i$$

□

Appendix B: Tables and Figures

Figure 2.6.1: The Distribution of V's Expected Wealth ($\bar{w}_V^i$): All Cases (N_{all})

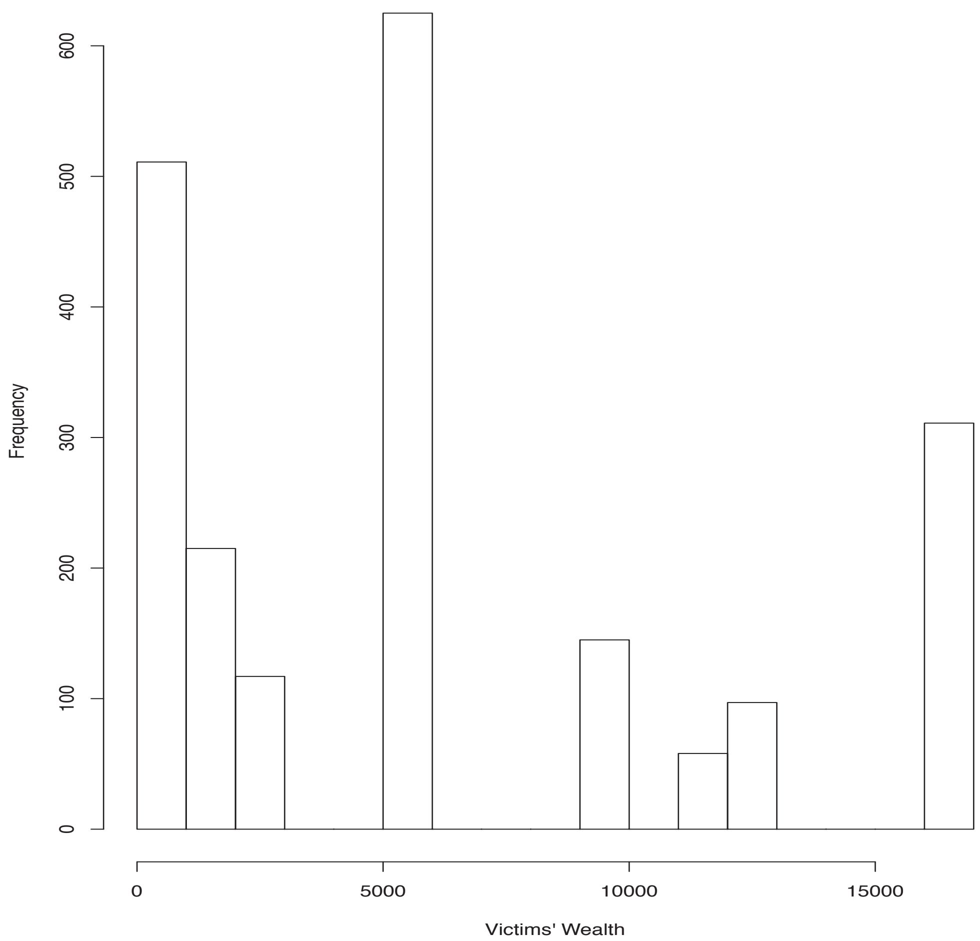

Table 2.9: Goodness-of-Fit: All Cases and Different Group (N_{all})

Moments	$E(P_S)$		$E(P_C \mid no\, S)$		$E(P(x^* = 1) \mid C)$	
Samples	Data (1)	Sim. (2)	Data (3)	Sim. (4)	Data (5)	Sim. (6)
Groups by victim-specific characteristics						
Female victim	.196	.15	.444	.444	.382	.432
		(2.9e-4)		(6.1e-4)		(6.8e-4)
Child victim	.161	.385	.398	0.32	.413	0.54
		(5.9e-4)		(1.6e-3)		(1.7e-3)
Unemployed victim	.148	.166	.454	.424	.457	.447
		(2.8e-4)		(5.7e-4)		(8.3e-4)
Victim of age 30–49	.168	.102	.463	.442	.415	.436
		(3.4e-4)		(6.8e-4)		(8.9e-4)
Groups by defendant-specific characteristics						
Female defendant	.214	.164	.437	.436	.187	.393
		(5.3e-4)		(1.2e-3)		(1.9e-3)
Defendant of age 25–29	.21	.128	.485	.462	.418	.438
		(3.2e-4)		(7.4e-4)		(1.2e-3)
Defendant of age 30–39	.156	.156	.508	.434	.4	.452
		(3.8e-4)		(6.9e-4)		(9.9e-4)
Defendant is a law enforcer or a government official	.143	.172	.5	.448	.5	.424
		(2.1e-3)		(4.1e-3)		(6.8e-3)
Defendant holds a college degree	.245	.175	.432	.418	.316	.424
		(4.8e-4)		(1.04e-3)		(1.4e-3)
Defendant holds a high school degree	.104	.156	.535	.439	.487	.442
		(3.1e-4)		(5.9e-4)		(9.4e-4)
Defendant has a criminal record	.132	.139	.454	.442	.444	.513
		(3.7e-4)		(5.7e-4)		(9.6e-4)

Note:
To simulate the model, 1'000 draws from the estimated distributions of w_V, w_D, a, b, m_V, m_D, and xp^h are taken. This procedure is repeated 100 times.
$E(P_S)$ denotes an expected probability to settle. $E(P_C \mid no\, S)$ reflects an expected probability to end up in court given no settlement. $E(P(x = 1 \mid C))$ defines an expected probability to get a real sentence once the case goes to court. For the data, $E(P_S)$, $E(P_C \mid no\, S)$, and $E(P(x = 1 \mid C))$ correspond to a frequency of observing $s = 1$, $c = 1$, and $x^* = 1$, respectively. In case of simulations, the values are computed based on the estimated distributions. Standard deviations are reported in parentheses.

Table 2.10: Estimation Results: "Car vs. Pedestrian" Cases (N_p)

Variable	Coefficient	St. Error
Victim's Wealth		
Intercept	88.62	87.484
SES_V:		
worker	21.31***	2.348
office worker	88.14***	1.157
top-manager	44.49***	2.849
entrepreneur	−12.02***	.074
budget office worker	92.03***	8.033
student	82.34**	32.502
welfare recipient	−3.21	4.265
retired	88.52***	0.534
other	90.67***	26.314
$gender_V$	27.72***	1.573
$child_V$	137.05**	2.058
age_V	45.71***	1.629
age_V^2	−0.53**	.267
Defendant's Wealth		
Intercept	22.72	25.329
SES_D:		
worker	66.19**	25.753
office worker	87.37***	.163
top-manager	36.3*	21.161
entrepreneur	88.59***	1.474
budget office worker	96.7***	3.779
student	105.1***	1.396
welfare recipient	25.8***	.408
retired	66.5***	.9
other	25.66	23.324
$gender_D$	21.34***	1.279
age_D	57.36***	.326
age_D^2	−0.81***	.113
edu_D	104.18***	15.686
$pcar_D$	1	None
Vindictiveness (a)		
Intercept	89.71***	24.232
w_V	0.19	30.175
Defendant's Disutility of Punishment (b)		
Intercept	73.21***	0.185
w_D	18.72	18.612
N		1055

Variable	Coefficient	St.Error
Non-Monetary Fighting Abilities		
Victim:		
Intercept	56.87***	2.119
$lawenf_V$	56.55**	23.13
Defendant:		
Intercept	8.81***	2.02
$lawenf_D$	12.57	19.829
Accident and Harm		
Number of		
dead victims	154.81***	26.143
victims with serious injuries	11.18	15.245
dead minor victims	82.52***	1.604
minor victims with serious injuries	166.56***	16.826
dead female victims	90.19***	1.903
female victims with serious injuries	−1.25***	4.739
dead minor female victims	28.28***	3.352
minor female victims with serious injuries	12.69	27.234
$drunk_D$	173.7***	.332
$drunk_V$	−34.22***	.378
$crimehist_D$	89.06***	.631
$admhist_D$	37.89***	.832
$record_D$	81.42***	27.426
Region-specific controls	Yes	
t	291.18***	.436
Underlying Distributions		
σ_V^w	12.91***	.246
σ_D^w	10.61***	1.977
σ_a	11.07***	.528
σ_b	4.1***	.189
σ_V^m	7.75***	.239
σ_D^m	67.83***	.331
σ_x	158.87***	12.441
$log(L)$	–1501.21	

Table 2.11: Goodness-of-Fit: "Car vs. Pedestrian" Cases (N_p)

Moments	$E(P_S)$		$E(P_C \mid no\, S)$		$E(P(x^* = 1) \mid C)$	
Samples	Data (1)	Sim. (2)	Data (3)	Sim. (4)	Data (5)	Sim. (6)
All cases:	.163	.123 (2.7e-4)	.399	.495 (1.3e-3)	.406	.453 (6.7e-4)
Pearson's χ^2 stat.	38.78					
Critical χ^2_3 ($\alpha = .99$)	9.21					
Groups by victim-specific characteristics						
Female victim	.173	.12 (4.3e-4)	.392	.5 (1.7e-3)	.41	.462 (9.8e-4)
Child victim	.187	.17 (8.8e-4)	.356	.418 (2.9e-3)	.46	.598 (2.2e-3)
Unemployed victim	.16	.12 (4.1e-4)	.384	.507 (1.7e-3)	.435	.476 (1.1e-3)
Victim of age 30–49	.13	.112 (6e-4)	.38	.515 (2.3e-3)	.39	.423 (1.3e-3)
Groups by defendant-specific characteristics						
Female defendant	.179	.156 (1.01e-3)	.344	.446 (4.3e-3)	.12	.329 (2.5e-3)
Defendant of age 25–29	.206	.126 (6.3e-4)	.398	.491 (2.6e-3)	.391	.459 (1.5e-3)
Defendant of age 30–39	.151	.123 (5.3e-4)	.4	.495 (1.9e-3)	.51	.499 (1.5e-3)
Defendant is a law enforcer or a government official	.167	.1 (2.7e-3)	.5	.571 (6e-3)	.6	.551 (6.8e-3)
Defendant holds a college degree	.195	.15 (6.8e-4)	.28	.461 (2.8e-3)	.283	.4 (1.8e-3)
Defendant holds a high school degree	.07	.1 (4.6e-4)	.463	.514 (2e-3)	.513	.455 (1.1e-3)
Defendant has a criminal record	.162	.08 (6.1e-4)	.446	.548 (2.1e-3)	.419	.577 (1.5e-3)

Note:
To simulate the model, 1'000 draws from the estimated distributions of w_V, w_D, a, b, m_V, m_D, and xp^h are taken. This procedure is repeated 100 times.
$E(P_S)$ denotes an expected probability to settle. $E(P_C \mid no\, S)$ reflects an expected probability to end up in court given no settlement. $E(P(x = 1 \mid C))$ defines an expected probability to get a real sentence once the case goes to court. For the data, $E(P_S)$, $E(P_C \mid no\, S)$, and $E(P(x = 1 \mid C))$ correspond to a frequency of observing $s = 1$, $c = 1$, and $x^* = 1$, respectively. In case of simulations, the values are computed based on the estimated distributions. Standard deviations are reported in parentheses.

Table 2.12: The Effects of Increasing D's Wealth: "Car vs. Pedestrian" Cases (N_p)

Moments	$E(P_S)$	N_S	$\bar{S}$
Wealth			
$w_D^1 = 728$	.01	10	1.28
	(5.2e-5)	(.055)	(.012)
$w_D^2 = 2'082$	.026	28	11
	(6.8e-5)	(.072)	(.063)
$w_D^3 = 2'250$	.028	30	12.87
	(8.2e-5)	(.087)	(.055)
$w_D^4 = 2'444$	.03	32	15.3
	(8.3e-5)	(.088)	(.079)
$w_D^5 = 2'816$	.035	37	20.49
	(5.1e-5)	(.053)	(.075)
$w_D^6 = 7'283$	.084	89	138.73
	(1.3e-4)	(.139)	(.348)
$w_D^7 = 19'405$	.178	187	702.79
	(9.5e-5)	(.1)	(.788)
$w_D^8 = 20'826$	.185	195	765.45
	(1e-4)	(.114)	(1.425)
$w_D^9 = 22'506$	.192	203	834.1
	(9.7e-5)	(.103)	(1.639)
$w_D^{10} = 27'145$	.21	221	993.45
	(9.2e-5)	(.097)	(1.569)
$w_D^{11} = 28'166$	.213	224	1'020.51
	(7.6e-5)	(.08)	(1.543)
$w_D^{12} = 41'653$	.237	250	**1'190.61**
	(5e-5)	(.053)	(1.864)
$w_D^{13} = 45'013$	.241	253	1'183.92
	(4.8e-5)	(.05)	(1.839)
$w_D^{14} = 48'891$	.243	256	1'162.33
	(3.7e-5)	(.039)	(2.149)
$w_D^{15} = 56'333$	.246	260	1'084.4
	(2.3e-5)	(.024)	(2.065)

Note:
To simulate the model, 1'000 draws from the estimated distributions of w_V, w_D, a, b, m_V, m_D, and xp^h are taken. This procedure is repeated 100 times. w_D^1–w_D^5 correspond to 0, .25, .5, .75, and 1 quantiles of $\bar{w}_D^i$'s estimated distribution; w_D^6–w_D^{10} and w_D^{11}–w_D^{15} reflect w_D^1–w_D^5 multiplied by 10 and 20, respectively. w_D is measured in rubles. Standard errors are reported in parentheses.

Table 2.13: The Effects of Increasing V's Wealth: All Cases (N_{all})

Moments	$E(P_S)$	$P(a > b)$	$P(S \in (-\pi_D^*, w_D])$
Wealth			
$w_V^1 = 13'099$	4.2e-3	0	0
	(2e-5)	(–)	(–)
$w_V^2 = 51'082$	1.1e-3	0	0
	(1.3e-5)	(–)	(–)
$w_V^3 = 130'995$	4.6e-4	0	0
	(6.5e-6)	(–)	(–)
$w_V^4 = 168'875$	3.6e-4	0	0
	(7.2e-6)	(–)	(–)
$w_V^5 = 261'991$	2.3e-4	0	0
	(5.1e-6)	(–)	(–)
$w_V^6 = 392'987$	1.6e-4	0	0
	(4.4e-6)	(–)	(–)
$w_V^7 = 510'826$	1.2e-4	0	0
	(3.8e-6)	(–)	(–)
$w_V^8 = 523'983$	1.2e-4	0	0
	(3.9e-6)	(–)	(–)
$w_V^9 = 1'021'652$	6.2e-5	.23	4.8e-8
	(2.5e-6)	(–)	(6.9e-8)
$w_V^{10} = 1'532'479$	4.1e-5	.23	5.3e-7
	(2e-6)	(–)	(2.3e-7)
$w_V^{11} = 1'688'750$	3.7e-5	.39	5.3e-7
	(2.2e-6)	(5.1e-5)	(2.4e-7)
$w_V^{12} = 2'043'305$	3e-5	.46	7.2e-7
	(2.1e-6)	(–)	(2.7e-7)
$w_V^{13} = 3'377'500$	1.7e-5	.72	1.2e-6
	(1.1e-6)	(–)	(3.7e-7)
$w_V^{14} = 5'066'250$	1e-5	.86	1.3e-6
	(9.5e-7)	(–)	(3.2e-7)
$w_V^{15} = 6'755'000$	6.9e-6	.95	1.4e-6
	(9.4e-7)	(–)	(4.1e-7)

Note:
To simulate the model, 1'000 draws from the estimated distributions of w_V, w_D, a, b, m_V, m_D, and xp^h are taken. This procedure is repeated 100 times. w_V^i, $i = \{1, ..., 15\}$ correspond to rescaled and sorted 0, .5, and 1 quantiles of $\bar{w}_V^i$'s estimated distribution (scaling factors are located between 1 and 40). $\bar{w}_V^i$ is measured in rubles. Standard errors are reported in parentheses.

Table 2.14: The Effects of Increasing V's Wealth: "Car vs. Pedestrian" Cases (N_p)

Moments	$E(P_S)$	$P(a > b)$	$P(S \in (-\pi^*_D, w_D])$
Wealth			
$w^1_V = 14'567$	1e-3	0	0
	(1.9e-5)	(–)	(–)
$w^2_V = 21'850$	6.9e-4	0	0
	(1.5e-5)	(–)	(–)
$w^3_V = 29'134$	5.2e-4	0	0
	(7.4e-6)	(–)	(–)
$w^4_V = 36'417$	4.3e-4	0	0
	(1e-5)	(–)	(–)
$w^5_D = 45'013$	3.6e-4	0	0
	(6.4e-6)	(–)	(–)
$w^6_D = 56'333$	2.9e-4	0	0
	(8.4e-6)	(–)	(–)
$w^7_D = 67'520$	2.4e-4	3.1e-5	0
	(5.9e-6)	(6.4e-6)	(–)
$w^8_D = 84'500$	1.9e-4	1	0
	(8.7e-6)	(–)	(–)
$w^9_D = 90'026$	1.8e-4	1	0
	(7.5e-6)	(–)	(–)
$w^{10}_D = 112'533$	1.4e-4	1	1.4e-6
	(3.4e-6)	(–)	(4.9e-7)
$w^{11}_D = 112'667$	1.3e-4	1	1.5e-6
	(6e-6)	(–)	(5.3e-7)
$w^{12}_D = 140'834$	1e-4	1	8.3e-6
	(3.5e-6)	(–)	(1.2e-6)

Note:
To simulate the model, 1'000 draws from the estimated distributions of w_V, w_D, a, b, m_V, m_D, and xp^h are taken. This procedure is repeated 100 times. D's wealth is fixed at $w^i_D = \min_i \{\bar{w}^i_D\}$. w^i_V, $i = \{1, ..., 14\}$ correspond to rescaled and sorted 0, .5, and 1 quantiles of $\bar{w}^i_V$'s estimated distribution (scaling factors are located between 1 and 50). $\bar{w}^i_V$ is measured in rubles. Standard errors are reported in parentheses.

Figure 2.6.2: The Distribution of D's Expected Wealth ($\bar{w}_D^i$): All Cases (N_{all})

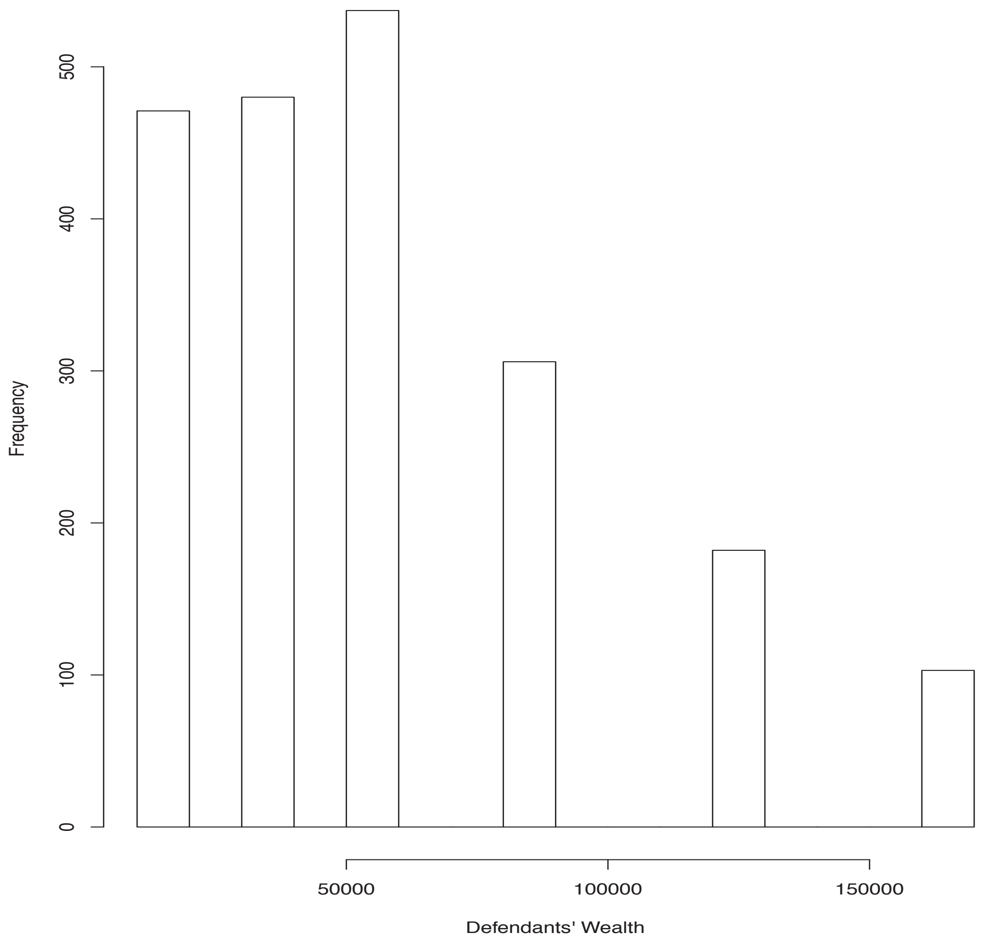

Figure 2.6.3: The Distribution of V's Expected Wealth ($\bar{w}_V^i$): "Car vs. Pedestrian" Cases (N_p)

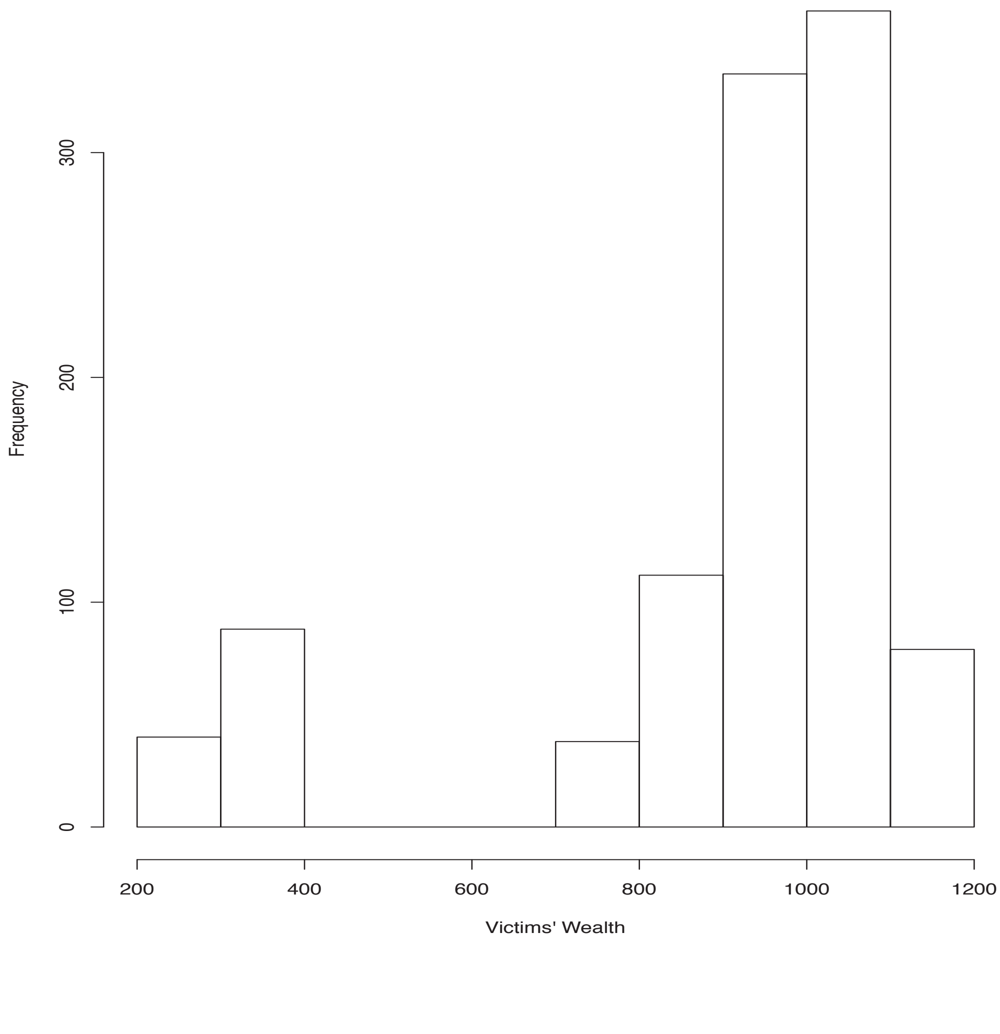

Figure 2.6.4: The Distribution of D's Expected Wealth ($\bar{w}_D^i$): "Car vs. Pedestrian" Cases (N_p)